

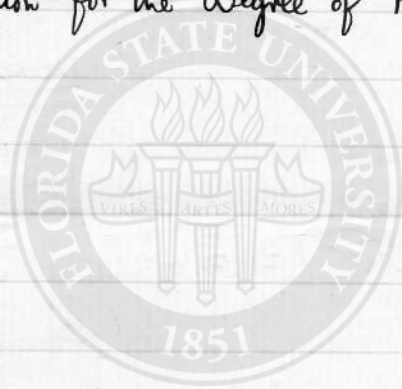
# QUANTUM MECHANICS

by

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Cambridge

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$$x\mu + t\sqrt{k^2 + \mu^2} = z$$

$$|t| > |x|$$

$$(x\mu + t\sqrt{k^2 + \mu^2})^2 - (t\mu + x\sqrt{k^2 + \mu^2})^2 = (x^2 + t^2)\mu^2 + (t^2 - x^2)(k^2 + \mu^2) = (t^2 - x^2)k^2 = \gamma^2$$

$$t\mu + x\sqrt{k^2 + \mu^2} = \pm\sqrt{z^2 - \gamma^2}$$

$$(t^2 - x^2)\mu = \pm t\sqrt{z^2 - \gamma^2} - xz$$

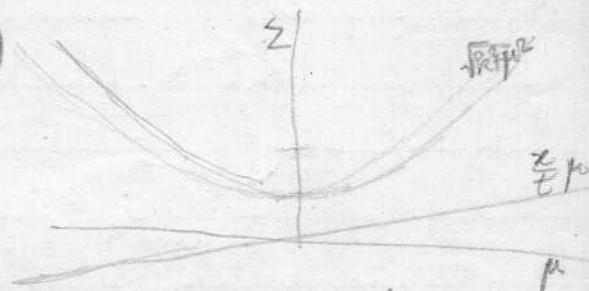
$$(t^2 - x^2)\sqrt{k^2 + \mu^2} = t z - x\sqrt{z^2 - \gamma^2}$$

$$\frac{\gamma^2}{k^2} d\mu = \left( \frac{\pm z}{\pm\sqrt{z^2 - \gamma^2}} - x \right) dz = \frac{dz}{\pm\sqrt{z^2 - \gamma^2}} (t z - x\sqrt{z^2 - \gamma^2})$$

$$\frac{\gamma^2}{k^2} \sqrt{k^2 + \mu^2} = t z - x\sqrt{z^2 - \gamma^2}$$

$$\frac{d\mu}{\sqrt{k^2 + \mu^2}} = \pm \frac{dz}{\sqrt{z^2 - \gamma^2}}$$

$$\int_{-\infty}^{\infty} e^{i(x\mu + t\sqrt{k^2 + \mu^2})} \frac{d\mu}{\sqrt{k^2 + \mu^2}} = 2 \int_{\gamma}^{\infty} e^{iz} \frac{dz}{\sqrt{z^2 - \gamma^2}} + \int_{-\infty}^{\gamma} e^{iz} \frac{dz}{-\sqrt{z^2 - \gamma^2}}$$



$$\text{Ans. } \psi = P(k^2(t^2 - x^2)) = P(\gamma^2)$$

$$\alpha > 0 \quad P(\alpha) = \int_{\alpha}^{\infty} e^{iz} \frac{dz}{\sqrt{z^2 - \alpha^2}} = \int_{\alpha}^{\infty} \frac{dz}{z} e^{iz} \left(1 - \frac{\alpha^2}{z^2}\right)^{-\frac{1}{2}}$$

$$\frac{d}{d\alpha} P(\alpha) e^{-i\alpha} = - \int_0^{\infty} e^{ix} \frac{dx}{\sqrt{x(x+2\alpha)}} \quad z - \alpha = x$$

$$\frac{d}{d\alpha} (P(\alpha) e^{-i\alpha}) = - \int_0^{\infty} e^{ix} \frac{dx}{x^{\frac{1}{2}}(x+2\alpha)^{\frac{3}{2}}} = - \int_{\alpha}^{\infty} e^{i(z-\alpha)} \frac{dz}{(z+\alpha)\sqrt{z^2 - \alpha^2}}$$

$$P'(\alpha) = i P(\alpha) - \int_{\alpha}^{\infty} e^{iz} \frac{dz}{(z+\alpha)\sqrt{z^2 - \alpha^2}} = \int_{\alpha}^{\infty} e^{iz} \frac{dz}{\sqrt{z^2 - \alpha^2}} \left(i - \frac{1}{z+\alpha}\right)$$

$$Q(\alpha) = P(\alpha) e^{-i\alpha}$$

$$Q'(\alpha) = - \int_0^{\infty} e^{ix} \frac{dx}{(x+2\alpha)\sqrt{x(x+2\alpha)}}$$

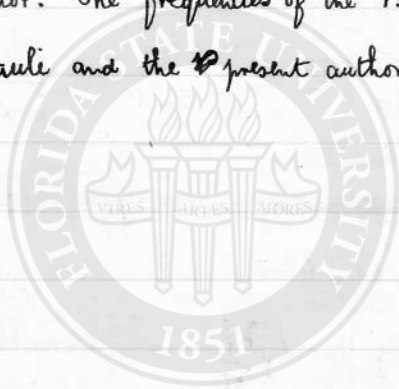
state

There is no direct way of specifying the ~~constants~~ of integration of a system by numbers.  
Such a specification must refer to a particular point of space time.

The code for specifying a state by numbers refers to a point of space time

## Preface.

The modern theory of quantum mechanics was introduced by Heisenberg in a paper in the Zeits. f. Phys. vol. 33, p. 879 (1925). The following dissertation is a development of this theory from a slightly different point of view from that of Heisenberg's paper. The theory has been developed independently from Heisenberg's original point of view by Born and Jordan (Zeits. f. Phys. vol. 34, p. 858, 1925<sup>5</sup>); Born, Heisenberg and Jordan (Zeits. f. Phys. vol. 35, p. 557, 1926) and Pauli (Zeits. f. Phys. vol. 36, p. 336, 1926). The general quantum conditions have been obtained independently by Born, Heisenberg and Jordan, by Hramers (Physica vol. 5, p. 369, 1925) and by the present author. The frequencies of the Balmer lines ~~of~~ of hydrogen have been obtained independently by Pauli and the ~~present~~ present author.



Darac  
↳

$$\textcircled{1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2 + \mu^2}} e^{i(x\mu + t\sqrt{k^2 + \mu^2})} d\mu = \int_{-\infty}^{\infty} \frac{1}{\sqrt{k'^2 + \mu'^2}} e^{i(x'\mu' + t'\sqrt{k'^2 + \mu'^2})} d\mu'$$

2

$$\frac{d\mu}{\sqrt{k^2 + \mu^2}} = \frac{(l + \frac{m\mu'}{\sqrt{k'^2 + \mu'^2}}) d\mu'}{m\mu' + l\sqrt{k'^2 + \mu'^2}} = \frac{d\mu'}{\sqrt{k'^2 + \mu'^2}}$$

$$\log(\mu + \sqrt{k^2 + \mu^2}) = \log(\mu' + \sqrt{k'^2 + \mu'^2}) + \text{const.}$$

$$\begin{aligned} (\mu + \sqrt{k^2 + \mu^2}) &= c(\mu' + \sqrt{k'^2 + \mu'^2}) \\ &= (l+m)(\mu' + \sqrt{k'^2 + \mu'^2}) \end{aligned}$$

$$x\mu + t\sqrt{k^2 + \mu^2} = x'\mu' + t'\sqrt{k'^2 + \mu'^2}$$

$$d\mu \left( x + \frac{t\mu}{\sqrt{k^2 + \mu^2}} \right) = d\mu' \left( x' + \frac{t'\mu'}{\sqrt{k'^2 + \mu'^2}} \right)$$

$$\psi_2' = \psi_2$$

$$\psi_1 = -i \frac{\partial \psi_2}{\partial t} = -i \left( \frac{\partial \psi_2'}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial \psi_2'}{\partial x'} \frac{\partial x'}{\partial t} \right) = -i \left( l \frac{\partial \psi_2'}{\partial t'} + m \frac{\partial \psi_2'}{\partial x'} \right) = l\psi_1' - i m \frac{\partial \psi_2'}{\partial x'}$$

General solution

$$\psi = a\psi_1 + b\psi_2$$

$$\psi' = a'\psi_1' + b'\psi_2'$$

$\psi_1, \frac{\partial \psi_1}{\partial t}$  at any point depend only on  $\psi_2, \frac{\partial \psi_2}{\partial t}$  in neighbourhood of that point.

$$\psi' = a' \left( l\psi_1 + im \frac{\partial \psi_2}{\partial x} \right) + b'\psi_2 = a_1\psi + a_2 \frac{\partial \psi}{\partial t} + a_3 \frac{\partial \psi}{\partial x} + a_4 \frac{\partial^2 \psi}{\partial x \partial t} + \dots$$

$$= -ia'l \frac{\partial \psi_2}{\partial t} + ia'm \frac{\partial \psi_2}{\partial x} + b'\psi_2 = a_1 \left( -ia \frac{\partial \psi_2}{\partial t} + b\psi_2 \right) + a_2 \frac{\partial}{\partial t} \left( -ia \frac{\partial \psi_2}{\partial t} + b\psi_2 \right) + a_3 \frac{\partial}{\partial x} \left( -ia \frac{\partial \psi_2}{\partial t} + b\psi_2 \right) + \dots$$

$$\text{If } a=0 \text{ and } a'=0 \quad b' = a_1 b \quad a_2 = a_3 = \dots = 0.$$

$$r_{\text{out}} \mu + \sqrt{k^2 + \mu^2} = \theta$$

$$-k + \sqrt{k^2 + \mu^2} = \frac{k^2}{\theta}$$

$$d\mu \left( 1 + \frac{\mu}{\sqrt{k^2 + \mu^2}} \right) = d\theta \quad \frac{d\theta}{\theta} = \frac{d\mu}{\sqrt{k^2 + \mu^2}}$$

$$\begin{aligned} \mu &= -\theta \\ \sqrt{k^2 + \mu^2} &= \mu \left( 1 + \frac{k^2}{\mu^2} \right)^{\frac{1}{2}} \\ &= \mu + \frac{k^2}{2\mu} + \dots \end{aligned}$$

$$x\mu + t\sqrt{k^2 + \mu^2} = \frac{x+t}{2} (\mu + \sqrt{k^2 + \mu^2}) + \frac{x-t}{2} (\mu - \sqrt{k^2 + \mu^2}) = \frac{x+t}{2} \theta - \frac{x-t}{2} \frac{k^2}{\theta}$$

$$\int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{k^2 + \mu^2}} e^{i(x\mu + t\sqrt{k^2 + \mu^2})} = \int_0^{\infty} \frac{d\theta}{\theta} e^{\frac{i}{2}[(x+t)\theta + (x-t)\frac{k^2}{\theta}]} = \int_0^{\infty} \frac{d\theta'}{\theta'} e^{\frac{i}{2}[\theta' + (x-t)\frac{k^2}{\theta'}]}$$

$$\theta' = (x+t)\theta$$

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$$\psi_1 = \int_{-\infty}^{\infty} \left[ e^{i(x\mu + t\sqrt{k^2 + \mu^2})} + e^{i(x\mu - t\sqrt{k^2 + \mu^2})} \right] d\mu$$

$$\psi_2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2 + \mu^2}} \left[ e^{i(x\mu + t\sqrt{k^2 + \mu^2})} - e^{i(x\mu - t\sqrt{k^2 + \mu^2})} \right] d\mu$$

$$-i \frac{\partial \psi_1}{\partial t} = \int_{-\infty}^{\infty} \sqrt{k^2 + \mu^2} \left[ e^{i(x\mu + t\sqrt{k^2 + \mu^2})} - e^{i(x\mu - t\sqrt{k^2 + \mu^2})} \right] d\mu$$

$$-i \frac{\partial \psi_2}{\partial t} = \int_{-\infty}^{\infty} \left[ e^{i(x\mu + t\sqrt{k^2 + \mu^2})} + e^{i(x\mu - t\sqrt{k^2 + \mu^2})} \right] d\mu$$

$$x' = lx + mt$$

$$l^2 - m^2 = 1$$

$$x'^2 - t'^2 = x^2 - t^2$$

$$t' = lt + mx$$

(1)

(2)

$$x'\mu' + t'\sqrt{k^2 + \mu'^2} = x\mu + t\sqrt{k^2 + \mu^2}$$

$$\text{if } l\mu' + m\sqrt{k^2 + \mu'^2} = \mu$$

$$l\mu - m\sqrt{k^2 + \mu^2} = \mu'$$

$$m\mu' + l\sqrt{k^2 + \mu'^2} = \sqrt{k^2 + \mu^2}$$

$$d\mu = \left( l + \frac{m\mu'}{\sqrt{k^2 + \mu'^2}} \right) d\mu'$$

(1)

$$\int_{-\infty}^{\infty} e^{i(x\mu + t\sqrt{k^2 + \mu^2})} d\mu = \int_{-\infty}^{\infty} e^{i(x'\mu' + t'\sqrt{k^2 + \mu'^2})} \left( l + \frac{m\mu'}{\sqrt{k^2 + \mu'^2}} \right) d\mu'$$

$$= l \int_{-\infty}^{\infty} e^{i(x'\mu' + t'\sqrt{k^2 + \mu'^2})} d\mu' + im \frac{\partial}{\partial x'} \int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2 + \mu'^2}} e^{i(x'\mu' + t'\sqrt{k^2 + \mu'^2})} d\mu'$$

$$(2) \int_{-\infty}^{\infty} e^{i(x\mu - t\sqrt{k^2 + \mu^2})} d\mu = \int_{-\infty}^{\infty} e^{i(x'\mu' - t'\sqrt{k^2 + \mu'^2})} \left( l - \frac{m\mu'}{\sqrt{k^2 + \mu'^2}} \right) d\mu'$$

$$= l \int_{-\infty}^{\infty} e^{i(x'\mu' - t'\sqrt{k^2 + \mu'^2})} d\mu' + im \frac{\partial}{\partial x'} \int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2 + \mu'^2}} e^{i(x'\mu' - t'\sqrt{k^2 + \mu'^2})} d\mu'$$

$$\psi_1 = l\psi_1' - im \frac{\partial}{\partial x'} \psi_2'$$

$$\psi_1' = l\psi_1 + im \frac{\partial}{\partial x} \psi_2$$

$$\text{given } \psi_1 = l(l\psi_1 + im \frac{\partial \psi_2}{\partial x}) - im \left( \frac{\partial \psi_2}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial \psi_2}{\partial t} \frac{\partial t}{\partial x'} \right)$$

$$= l^2 \psi_1 - m^2 \psi_1$$

§1. Systems involving  $t$  explicitly  $t$  a  $q$ -number

§2.  $x, y, z, t$  in  $p, q$  etc.  $t = t - \frac{1}{c}$  etc.

§3. Systems in a classical field.  
Q.T. gives meaning to P.B. Emphasizes fundamental nature of  $q$ -variables.

Any attempt to extend the applicability of Q.T. must be preceded by a formulation of the classical problem in terms of canonical variables & P.B.'s.

# Relativistic Quantum Mechanics of Moving Systems, with an Application to Compton Scattering.

## §1. Introduction

The new quantum mechanics introduced by Heisenberg and since developed from different points of view by <sup>various authors</sup> takes its simplest form if we assume <sup>namely</sup> that the dynamical variables are members of a special type (called  $q$ -numbers to distinguish them from ordinary or  $c$ -numbers) which <sup>obey all the</sup> ordinary algebraic laws except that they do not satisfy the commutative law of multiplication, and satisfy instead of this, the relations

$$q_1 q_2 - q_2 q_1 = i\hbar [q_1, q_2] \quad (1)$$
$$p_i q_j - q_j p_i = i\hbar \delta_{ij} \quad (2)$$

where the  $p$ 's and  $q$ 's are a set of canonical variables, and where  $\hbar$  is a  $c$ -number, equal to  $(2\pi)^{-1}$  times the usual Planck's constant, and  $[x, y]$  is a quantity which is closely analogous to the Poisson bracket expression of  $x$  and  $y$  in the classical theory, and is, in fact, unequal to 0,  $\infty$  when  $x$  and  $y$  are <sup>two of a set of</sup> canonical variables according to whether they are conjugate or not.

Equations (1) may be regarded as replacing the commutative law of the classical theory, and one can with their help, build up a complete algebraic theory of quantities that are analytic functions of a set of canonical variables. <sup>(1B)</sup> All the important equations of <sup>classical</sup> dynamics can be written in a form in which all differential coefficients have been replaced by P.B.'s, and the same then be taken over directly into the new mechanics. <sup>It will be observed that the</sup>

The notion of canonical variables plays a very fundamental part in the present theory. It is absolutely necessary that all the dynamical variables of the system under consideration shall be functions of a set of variables which are assumed to be canonical. Any attempt to extend the domain of the present quantum mechanics must be preceded by the introduction of canonical variables into the corresponding classical theory, <sup>with</sup> and a reformulation of <sup>classical</sup> this theory <sup>with</sup> in terms of P.B.'s instead of diff. eqs. \*1A

§2. Quantum Theory  
Consider a dynamical system for which the Hamiltonian  $H$  involves the time explicitly. <sup>It is known that and</sup> On the classical theory one may solve the problem by considering the time  $t$  to be an extra co-ordinate of <sup>the</sup> system. The principle of relativity demands that the time shall be on the same footing as the other variables, and it must therefore be a  $q$ -number.

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + c\right)\psi = 0$$

$$y = xu \quad u = \frac{y}{x} \\ x = x$$

$$\left(\frac{\partial}{\partial x}\right)_y = \left(\frac{\partial}{\partial x}\right)_u \left(\frac{\partial x}{\partial u}\right) + \frac{\partial}{\partial u} \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \frac{y}{x^2} \frac{\partial}{\partial u} = \frac{\partial}{\partial x} - \frac{u}{x} \frac{\partial}{\partial u}$$

$$\frac{\partial}{\partial y} = \frac{1}{x} \frac{\partial}{\partial u} = \frac{1}{x} \frac{\partial}{\partial u}$$

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \frac{u}{x} \frac{\partial}{\partial u} - \frac{u}{x} \frac{\partial}{\partial u} \frac{\partial}{\partial x} + \frac{u}{x} \frac{\partial}{\partial u} \frac{u}{x} \frac{\partial}{\partial u} - \frac{1}{x^2} \frac{\partial^2}{\partial u^2}$$

$$= \frac{\partial^2}{\partial x^2} - \frac{1}{x^2} \frac{\partial^2}{\partial u^2} + \frac{u}{x^2} \left( \frac{\partial}{\partial u} + u \frac{\partial^2}{\partial u^2} \right) - u \left( -\frac{1}{x^2} \frac{\partial}{\partial u} + \frac{1}{x} \frac{\partial^2}{\partial u \partial x} \right)$$

$$= \frac{\partial^2}{\partial x^2} - \frac{1}{x^2} \frac{\partial^2}{\partial u^2} + \frac{2u}{x^2} \frac{\partial}{\partial u} + \frac{u^2}{x^2} \frac{\partial^2}{\partial u^2} - \frac{2u}{x} \frac{\partial^2}{\partial u \partial x}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + c\right)X = 0$$

$$\left(\frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r}\right) + rc\right)X = 0$$

$$\phi = r^u X$$

$$u(u-1)r^{u-2}X + 2ur^{u-1} \frac{\partial X}{\partial r} + r^u \frac{\partial^2 X}{\partial r^2} + \frac{1}{r} \left( ur^{u-1}X + r^u \frac{\partial X}{\partial r} \right) + cr^u X = 0$$

$$u = -\frac{1}{2}$$

$$\frac{3}{4} \frac{1}{r^2} X + \frac{\partial^2 X}{\partial r^2} - \frac{1}{2} \frac{1}{r^2} X + cX = 0$$

$$\frac{\partial^2 X}{\partial r^2} + \frac{1}{4r^2} X + cX = 0$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = k\psi$$

$$\psi = f(x)g(y)$$

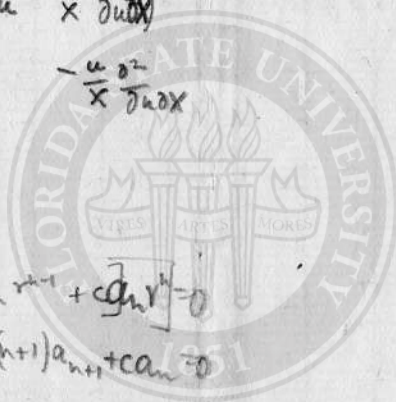
$$f'(x)g'(y) = k f(x)g(y)$$

$$\frac{f'(x)}{f(x)} = k$$

$$\frac{g'(y)}{g(y)} = \frac{k}{l}$$

$$f(x) = e^{lx}$$

$$\psi = e^{(lx + \frac{k}{l}y)}$$



## §2. Algebraic Axioms.

The branch of mathematics that is now to be developed deals with magnitudes of a kind that one cannot specify explicitly. These magnitudes can be ~~no~~ denoted by algebraic symbols and used analytically according to well-defined axioms, which are the same as those of ordinary algebra except for the commutative law of multiplication, which is not in general valid. We shall call these magnitudes *q-numbers*, to distinguish them from the numbers of ordinary mathematics, which will be called *c-numbers*, while the word *number* alone will be used to denote either a *q-number* or a *c-number*. Both *q-numbers* and *c-numbers* may appear together in the same piece of analysis, and even in the same equation. One cannot represent *q-numbers* in the decimal notation, and one cannot say that one *q-number* is greater or less than another.

If  $x$  and  $y$  are two *q-numbers*, ~~or~~ they are assumed to have a sum  $x+y$  and two products,  $xy$  and  $yx$ , which are not in general equal. They are assumed to satisfy all the laws of ordinary algebra except the commutative law of multiplication, i.e. if  $z$  is another *q-number*

$$x+y = y+x \quad (1.11)$$

$$(x+y)+z = x+(y+z) \quad (1.12)$$

$$(xy)z = x(yz) \quad (1.13)$$

$$x(y+z) = xy+xz, \quad (x+y)z = xz+yz. \quad (1.14)$$

and if

$$xy = 0$$

either

$$x=0, \text{ or } y=0;$$

but in general

$$xy \neq yx.$$

} (1.15)

In the special case when  $xy$  does equal  $yx$ , we shall say that  $x$  commutes with  $y$ .

If  $x$  and  $y$  are one *q-number* and one *c-number*, they are also assumed to have a sum  $x+y$  and a product  $xy$  or  $yx$ , these two quantities being <sup>assumed to be</sup> now equal, (i.e. a *q-number* commutes with a *c-number*). The algebraic axioms (1.11)–(1.15) are assumed to be still true when  $x$ ,  $y$  and  $z$  are some of them *q-numbers* and the others *c-numbers*. One can <sup>of course</sup> use *c-numbers* for the counting of *q-numbers*, e.g. one can put

$$\psi(x) = \int \psi(x') \psi(x'') dx$$

$$\psi(x) = \int \psi(x') \psi(x'') dx$$

Phase variable  $j = 1, 2$

$$\psi_1(x) = \cos vx$$

$$\psi_2(x) = \sin vx$$

$\psi(x)$  real.

$$\int_0^{2\pi} \cos ax da = \delta(a)$$

$$\int_0^{2\pi} \sin ax da = \frac{1}{x}$$

$$\int_0^{2\pi} a \cos ax da = -\frac{1}{x^2}$$

$$\psi(x) \text{ complex } \psi(x) = e^{ivx}$$

$$\psi(x'') = \int \psi(x') \psi'(x'') dx' = \int \psi(x') \cos vx'' dx'$$

$$= \int \psi(x') \cos vx'' dx' = \int \frac{\psi(x') \cos vx''}{x' - x''} dx' + \psi \delta(x' - x'')$$

$$= \delta(x' - x'') + \frac{\delta'(x' - x'')}{x' - x''}$$

$$\begin{aligned} \psi(x' - x'') &= \psi \delta(x' - x'') + \frac{\delta'(x' - x'')}{x' - x''} \\ &= (x' - x'') \psi(x' - x'') - \frac{1}{x' - x''} \end{aligned}$$

$$\psi(x' - x'') = \int \cos vx' \cdot x' \cdot \cos vx'' dx'$$

$$\sum_n \left[ \psi_n \frac{\partial \bar{\psi}_n}{\partial t} - \frac{\partial \psi_n}{\partial t} \bar{\psi}_n \right] \left[ \psi_n' \frac{\partial \bar{\psi}_n'}{\partial t} - \frac{\partial \psi_n'}{\partial t} \bar{\psi}_n' \right] - \text{same thing with dashed + undashed reversed}$$

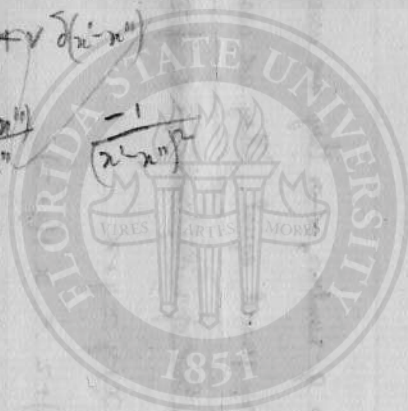
$$= -\delta(x - x') \frac{\partial \psi_n}{\partial t} \frac{\partial \bar{\psi}_n'}{\partial t}$$

$$\sum_n \bar{\psi}_n \psi_n' = \delta(x - x')$$

$$+ \sum_n \frac{\partial \bar{\psi}_n}{\partial t} \psi_n' \left[ \psi_n \frac{\partial \bar{\psi}_n'}{\partial t} - \frac{\partial \psi_n}{\partial t} \bar{\psi}_n' \right] \sum_n \frac{\partial \bar{\psi}_n}{\partial t} \psi_n' + \sum_n \bar{\psi}_n \frac{\partial \psi_n'}{\partial t} = 0$$

$$+ \sum_n \bar{\psi}_n \frac{\partial \psi_n'}{\partial t} \frac{\partial \bar{\psi}_n}{\partial t} \bar{\psi}_n' - \text{reverse}$$

$$- \sum_n \frac{\partial \bar{\psi}_n}{\partial t} \frac{\partial \psi_n'}{\partial t} \psi_n \bar{\psi}_n' + \text{reverse}$$



from which one can easily verify that  $x, y, z, p_x, p_y, p_z$  are canonical when it is assumed that the  $\xi$ 's and  $\eta$ 's are canonical. <sup>(35)</sup> These components  $\xi$  equations and also the most convenient ones for evaluating the amplitudes of the various components of vibration, since they give at once (This method shows that our previous  $\theta$  and  $\phi$  do commute) and hence that the  $x, y, z, p_x, p_y, p_z$  are canonical. Equations (17) are equivalent to equations (15) and from eqns (16) and (17) eqns (14) can easily be deduced; so that this transformation of the present  $\xi$  is the same as the ~~one of the~~ <sup>previous</sup> one (and proves that the previous  $\theta$  and  $\phi$  commute).

The value of  $z$  in terms of the new variables is given by

$$\begin{aligned} z m_z &= (p_x m_y - m_y p_x) = -\frac{1}{2}(x+iy)(m_x - im_y) - \frac{1}{2}(x-iy)(m_x + im_y) \\ &= \frac{1}{2} i r k^{-\frac{1}{2}} \{ (\xi_1^2 - \eta_1^2) \xi_2 + (\xi_2^2 - \eta_2^2) \xi_1 \} k^{-\frac{1}{2}} \\ &= \frac{1}{2} i r k^{-\frac{1}{2}} \{ \xi_1 \xi_2 (\xi_1 \eta_1 + \xi_2 \eta_2) - \\ z &= \frac{1}{2} i r k^{-\frac{1}{2}} (\xi_1 \xi_2 + \eta_1 \eta_2) k^{-\frac{1}{2}} \end{aligned}$$

The formulas (16) and (17) may be written in terms of the  $k, m_z, \theta, \phi$

$$\left. \begin{aligned} x+iy &= -\frac{1}{2} r \left\{ \frac{(k+\eta_2 - \frac{1}{2}i)(k-\eta_1 - \frac{1}{2}i)}{k^{\frac{1}{2}}(k-\frac{1}{2})^{\frac{1}{2}}} e^{i(\theta+\phi)} + \frac{(k-\eta_2 + \frac{1}{2}i)(k+\eta_1 + \frac{1}{2}i)}{k^{\frac{1}{2}}(k+\frac{1}{2})^{\frac{1}{2}}} e^{i(-\theta+\phi)} \right\} \\ x-iy &= \frac{1}{2} r \left\{ \frac{(k-\eta_2 - \frac{1}{2}i)(k+\eta_1 - \frac{1}{2}i)}{k^{\frac{1}{2}}(k-\frac{1}{2})^{\frac{1}{2}}} e^{i\theta} + \frac{(k+\eta_2 + \frac{1}{2}i)(k-\eta_1 + \frac{1}{2}i)}{k^{\frac{1}{2}}(k+\frac{1}{2})^{\frac{1}{2}}} e^{i(\theta+\phi)} \right\} \\ z &= \frac{1}{2} r \left\{ \frac{(k+\eta_1 - \frac{1}{2}i)(k-\eta_2 - \frac{1}{2}i)}{k^{\frac{1}{2}}(k-\frac{1}{2})^{\frac{1}{2}}} e^{i\theta} - \frac{(k+\eta_2 + \frac{1}{2}i)(k-\eta_1 + \frac{1}{2}i)}{k^{\frac{1}{2}}(k+\frac{1}{2})^{\frac{1}{2}}} e^{i\theta} \right\} \end{aligned} \right\} \quad (36)$$

From these relations the amplitudes of the different vibrations can be obtained by substituting for  $k$  and  $m_z$  in the coefficients their quantized values. (See 58)

For the case of  $m_z = c$  there are  $c$  electrons there are no corresponding variables  $\xi$  and  $\eta$  which enable the transformation eqns to be put in a form corresponding to (16) & (17)

The simplification of eqns (35) is due to the fact that one can associate each component of vibration with the product of two of the  $\xi, \eta$  variables that are not conjugate. For systems of more than one electron there are too many components of vibration for this to be done, so that there are no equations corresponding to (35) for the formal system.

$$\psi = \int_{-\infty}^{\infty} \left[ e^{i(kx + t\sqrt{k^2 + \mu^2})} \left( a + \frac{b}{\sqrt{k^2 + \mu^2}} \right) + e^{i(kx - t\sqrt{k^2 + \mu^2})} \left( a - \frac{b}{\sqrt{k^2 + \mu^2}} \right) \right] d\mu$$

$$-i \frac{\partial \psi}{\partial t} = \int_{-\infty}^{\infty} \left[ e^{i(kx + t\sqrt{k^2 + \mu^2})} (a\sqrt{k^2 + \mu^2} + b) + e^{i(kx - t\sqrt{k^2 + \mu^2})} (-a\sqrt{k^2 + \mu^2} + b) \right] d\mu$$

$$x' = \ell x + mt \quad x'^2 - t'^2 = (\ell^2 - m^2)(x^2 - t^2) = x^2 - t^2 \quad \ell^2 - m^2 = 1 \quad \ell > 0$$

$$t' = \ell t + mx$$

$$\psi' = \int_{-\infty}^{\infty} \left[ e^{i(x'\mu' + t'\sqrt{k'^2 + \mu'^2})} \left( a' + \frac{b'}{\sqrt{k'^2 + \mu'^2}} \right) + e^{i(x'\mu' - t'\sqrt{k'^2 + \mu'^2})} \left( a' - \frac{b'}{\sqrt{k'^2 + \mu'^2}} \right) \right] d\mu'$$

$$-x'\mu' + t'\sqrt{k'^2 + \mu'^2} = (\ell x + mt)\mu' + (\ell t + mx)\sqrt{k^2 + \mu^2}$$

$$= x(\ell\mu' + m\sqrt{k^2 + \mu^2}) + t(m\mu' + \ell\sqrt{k^2 + \mu^2}) = x\mu' + t\sqrt{k^2 + \mu^2}$$

$$\ell\mu' + m\sqrt{k^2 + \mu^2} = \mu$$

$$\ell\mu - m\sqrt{k^2 + \mu^2} = \mu'$$

$$m^2(k^2 + \mu^2) = (\mu - \ell\mu')^2 = \mu^2 + \ell^2\mu'^2 - 2\ell\mu\mu'$$

$$\mu^2 + \mu'^2 - 2\ell\mu\mu' + (\ell^2 - 1)k^2 = 0$$

$$\cancel{k^2 + \mu^2} = \cancel{\mu'^2}$$

$$m\mu' + \ell\sqrt{k^2 + \mu^2} = \sqrt{k^2 + \mu'^2}$$

$$m^4\mu'^4 + \ell^4(k^2 + \mu^2)^2 + (k^2 + \mu'^2)^2 - 2m^2\mu'^2\ell^2(k^2 + \mu^2) - 2m^2\mu'^2(k^2 + \mu'^2) - 2\ell^2(k^2 + \mu^2)(k^2 + \mu'^2) = 0$$

$$\left. \begin{aligned} &\mu'^4(m^4 + \ell^4 - 2m^2\ell^2) + \mu^4 - 2\mu^2\mu'^2(m^2 + \ell^2) \\ &+ 2\mu^2(\ell^4k^2 - m^2\ell^2k^2 - m^2k^2 - \ell^2k^2) + 2\mu^2(k^2 - \ell^2k^2) \\ &+ \ell^4k^4 + k^4 - 2\ell^2k^4 \end{aligned} \right\} = 0$$

$$\mu'^4 + \mu^4 - 2\mu^2\mu'^2(\ell^2 - 1) - 2\mu^2k^2m^2 - 2\mu^2k^2m^2 + k^4m^4 = 0$$

$$(\mu'^2 + \mu^2 - m^2k^2)^2 - 4\ell^2\mu^2\mu'^2 = 0$$

$$\ell\mu' + m\sqrt{k^2 + \mu^2} = \ell\mu = \mu' + m\sqrt{k^2 + \mu^2}$$

$$m\mu' + \ell\sqrt{k^2 + \mu^2} = \sqrt{k^2 + \mu'^2}$$

$$d\mu = \left( \ell + \frac{m\mu'}{\sqrt{k^2 + \mu^2}} \right) d\mu'$$

$$(a\sqrt{k^2 + \mu^2} + b)d\mu = \left[ \frac{a}{\ell} (m\mu' + \ell\sqrt{k^2 + \mu^2}) + b \right] \left( \ell + \frac{m\mu'}{\sqrt{k^2 + \mu^2}} \right) d\mu'$$

$$= \left[ \ell b + \frac{a}{\ell} m\mu' + \frac{a}{\ell} m\mu' + a\ell\sqrt{k^2 + \mu^2} + \frac{am^2\mu'^2}{\ell\sqrt{k^2 + \mu^2}} + \frac{bm\mu'}{\sqrt{k^2 + \mu^2}} \right] d\mu'$$

$$= \left[ \ell b - \frac{a}{\ell} k^2 + 2am\mu' + a\ell\sqrt{k^2 + \mu^2} + \frac{am^2}{\ell} \left( \frac{k^2 + \mu^2}{\sqrt{k^2 + \mu^2}} - \frac{k^2}{\sqrt{k^2 + \mu^2}} \right) + \frac{bm\mu'}{\sqrt{k^2 + \mu^2}} \right] d\mu'$$

$$\int_{-\infty}^{\infty} e^{i(x'\mu' + t'\sqrt{k'^2 + \mu'^2})} \left( x' + \frac{t'\mu'}{\sqrt{k'^2 + \mu'^2}} \right) d\mu' = 0$$

$$\int_{-\infty}^{\infty} e^{i(kx + t\sqrt{k^2 + \mu^2})} (a\sqrt{k^2 + \mu^2} + b) d\mu$$

$$= \int_{-\infty}^{\infty} e^{i(k'\mu' + t'\sqrt{k'^2 + \mu'^2})} \left[ \ell b - \frac{m\ell x'}{t'} + \frac{m\ell}{t'} \left( x' + \frac{t'\mu'}{\sqrt{k'^2 + \mu'^2}} \right) \right] d\mu'$$

$$\left[ \ell b - \frac{m\ell x'}{t'} + \frac{m\ell}{t'} \left( x' + \frac{t'\mu'}{\sqrt{k'^2 + \mu'^2}} \right) + 2am\mu' + \frac{a}{\ell} (\ell^2 + m^2) \left( \frac{k^2 + \mu^2}{\sqrt{k^2 + \mu^2}} - \frac{a\ell k^2}{\ell\sqrt{k^2 + \mu^2}} \right) \right] d\mu'$$

$$-i \frac{\partial \psi}{\partial t} =$$

### §5. On $q$ -numbers.

In the three preceding three §§ two unsatisfactory features will have been observed which will here be discussed. The first of these is that it is <sup>was</sup> continually necessary to postulate that a  $q$ -number exists that satisfies certain conditions, and the second is that the expression "every  $q$ -number" has frequently been used, which implies that there is a definite domain of <sup>all</sup>  $q$ -numbers. In the mathematics of  $c$ -numbers these difficulties do not occur because one can rigorously define the domain of all  $c$ -numbers, and then the statement that a  $c$ -number exists that satisfies certain conditions can be proved (if it is true) and need not be assumed. The state of affairs for  $q$ -numbers must be essentially different, owing to the undefinable nature of  $q$ -numbers.

One can safely assume that a  $q$ -number exists that satisfies certain conditions whenever these conditions do not lead to an inconsistency, since by a  $q$ -number one means only a dummy symbol appearing in the analysis satisfying these conditions. <sup>Also</sup> ~~Further~~, when one says that all  $q$ -numbers satisfy a certain condition, one needs this result to apply only to the  $q$ -numbers that one is actually dealing with in the problem. It would not do any harm if there was a  $q$ -number, entirely disconnected from all the  $q$ -numbers one is dealing with in the problem, that did not satisfy the condition, as the inconsistency, ~~and~~ which, of course, really exists, would not be "brought home" to one. Further, it might at a later stage in the problem actually be desirable to assume the existence of a  $q$ -number ~~that~~ that did not satisfy the condition. One would therefore have to consider the statement that all  $q$ -numbers satisfy <sup>the condition</sup> ~~as~~ applying only to those  $q$ -numbers with which one is dealing at the time. One is thus led to consider that the domain of all the  $q$ -numbers is elastic, and is liable at any time to be extended by fresh assumptions of the existence of  $q$ -numbers satisfying certain conditions, and that when one says that all  $q$ -numbers satisfy a certain condition, one means it to apply only to the then existing domain of  $q$ -numbers, and not to exclude the possibility of a later extension of the domain to  $q$ -numbers that do not satisfy the condition.

As an example of the necessity of this point of view, consider the definition of  $x^{\frac{1}{2}}$ . It is defined to commute with every number that commutes with  $x$ , and to satisfy the equation  $x^{\frac{1}{2}} x^{\frac{1}{2}} = x$ . Now from assumption (i) of §2 at the end of §2, there must be a number  $b$  such that

$$\psi_m \frac{\partial \bar{\psi}_n}{\partial t} - \bar{\psi}_n \frac{\partial \psi_m}{\partial t}$$

$$a_m = a_n \quad b_{11} = b_{22}$$

$$(ab)_{11} = a_{11}b_{11} + a_{22}b_{22} = a_{22}b_{22} + a_{11}b_{11} = (ba)_{22}$$

any imag.  $\psi_n$

$$\nabla^2 \psi - \frac{\partial \psi}{\partial t} = k^2 \psi$$

$$\mu = \sqrt{k^2 + \alpha^2 + \beta^2 + \gamma^2}$$

$$\psi = \int_{-\infty}^{\infty} \left[ e^{i(\alpha x + \beta y + \gamma z + \mu t)} + e^{i(\alpha x + \beta y + \gamma z - \mu t)} \right] d\alpha d\beta d\gamma$$

$$= \int_{-\infty}^{\infty} \left[ e^{iK \sin \theta \cos \phi + iK \sin \theta \sin \phi + iK \cos \theta} \right] K^2 dK \sin \theta d\theta d\phi$$

$$\alpha = K \sin \theta$$

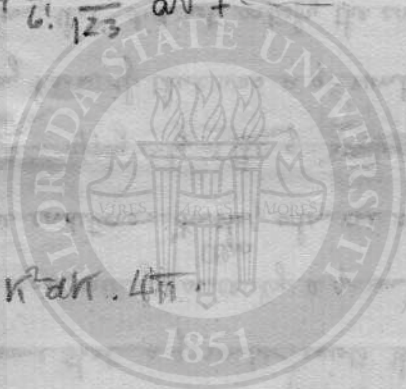
$$\beta = K \sin \theta \sin \phi$$

$$\gamma = K \cos \theta$$

$$\int_0^{2\pi} e^{ae^{i\phi} + be^{-i\phi}} d\phi = 2\pi \left[ \text{constant} + \frac{1}{2!} (ae^{i\phi} + be^{-i\phi})^2 + \frac{1}{4!} (ae^{i\phi} + be^{-i\phi})^4 + \frac{1}{6!} (ae^{i\phi} + be^{-i\phi})^6 + \frac{1}{8!} (ae^{i\phi} + be^{-i\phi})^8 + \dots \right]$$

$$= 2\pi \left[ 1 + \frac{1}{2!} 2ab + \frac{1}{4!} 6a^2b^2 + \frac{1}{6!} \frac{6 \cdot 5 \cdot 4}{123} a^3b^3 + \dots \right]$$

$$= 2\pi \sum_{r=0}^{\infty} \frac{1}{(2r)!} (ab)^r \frac{(2r)!}{r! r!}$$



Take  $y=0, z=0$

$$\psi = \int_{-\infty}^{\infty} e^{iK x \cos \theta} 2\pi \sin \theta K^2 dK \cdot 4\pi$$

analogues as closely as possible, it is desirable that we should be able to give a meaning to real and imaginary  $q$ -numbers. We can do this by defining the conjugate imaginary of an algebraic expression involving only real numbers and  $i$ , to be the number obtained when one writes  $-i$  for  $i$  and <sup>which they are assumed to be if they are the analogues of real classical quantities</sup> reverses the order of the factors of all products. Thus if  $p$  and  $q$  are real, the conjugate imaginary of  $(qp - pq)$  is  $pq - qp$ , equal to  $-(qp - pq)$ , and hence if

$$qp - pq = ih,$$

$h$  is real. It is to be doubted, though, whether the notion of real and imaginary  $q$ -numbers can be will survive in the ultimate theory, since, if one <sup>could</sup> ~~give meaning~~ distinguish between real and imaginary  $q$ -numbers, one could define a positive  $q$ -number to be one whose square roots <sup>are</sup> were real and a negative  $q$ -number to be one whose square roots <sup>are</sup> were imaginary, and one could then give a meaning to one  $q$ -number being greater or less than another. This appears to be carrying the analogy with  $c$ -numbers too far. This difficulty is probably connected with the difficulties in the theory of square roots.

We have been able to develop the pure mathematics of  $q$ -numbers thus far without any reference to the <sup>the</sup> notion of a limit of a sequence of  $q$ -numbers. It might appear to be desirable to carry through the whole <sup>of the</sup> theory without the use of limits. To attempt to do so at the present time, however, would completely destroy the very close analogy that exists between the present theory and classical dynamics, owing to the fact that limits are ~~so~~ repeatedly used in classical dynamics, particularly through the sums of Fourier series. We shall therefore use limits freely in the subsequent work in analogy to the classical theory, although it does not appear to be possible to give a general definition of the limit of a sequence of  $q$ -numbers.

$$p_{mn} = X(x_1, m) \psi(x_1, n) + X(x_2, n) \psi(x_2, m)$$

at time  $t$

Wave fun for state  $n$  is  $\psi(x_i, n)$   $i=1$  or  $2$

$$\psi(x_i) = \int_{x-\epsilon}^{x+\epsilon} [B(x_i, x'_1) \psi_t(x'_1) + B(x_i, x'_2) \psi_t(x'_2)] dx'$$

where

$$B(x_j, x'_k) = e^{2\pi i H(x_j, x'_k) \tau / \hbar}$$

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi$$

$$\psi_1 = P(t^2 - x^2) \quad \psi_2 = \psi_1 \frac{\partial \psi_1}{\partial t}$$

$$\left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \psi_1 = k^2 \psi_1$$

$$i\hbar \frac{\partial \psi_1}{\partial t} = H_{11} \psi_1 + H_{12} \psi_2$$

$$2 \frac{\partial \psi_2}{\partial t} = -\hbar^2 \frac{\partial^2 \psi_1}{\partial x^2} = -\hbar^2 \left( k^2 + \frac{\partial^2}{\partial x^2} \right) \psi_1$$

$$= H_{21} \psi_1 + H_{22} \psi_2$$

$$\begin{cases} H_{11} = 0 & H_{12} = 1 \\ H_{21} = -\hbar^2 \left( k^2 + \frac{\partial^2}{\partial x^2} \right) & H_{22} = 0 \end{cases}$$

$$\psi_1' = \psi_1$$

$$\psi_2' = \frac{\partial \psi_1}{\partial t} = \ell \psi_2 + m \frac{\partial \psi_1}{\partial x}$$

$$H = \begin{pmatrix} 0 & 1 \\ -\hbar^2 \left( k^2 + \frac{\partial^2}{\partial x^2} \right) & 0 \end{pmatrix}$$

Variable  $j=(1,2)$  commutes with  $x$  but not with  $H$ .

Does it commute with  $\frac{da}{dt} = [x, H]$

$$[j, [x, H]] = [x, [j, H]] + [H, [x, j]]$$

Find relation between  $k_n$  and  $x$

$$\text{To find } \sqrt{a + b \frac{\partial^2}{\partial x^2}} = A(x)$$

$$(a + b \frac{\partial^2}{\partial x^2}) \psi(x) = \lambda \psi(x)$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\lambda - a}{b} \psi$$

$$\frac{a - \lambda}{b} = k^2 \quad \lambda = a - b k^2$$

$$\psi = e^{\pm i \sqrt{\frac{a - \lambda}{b}} x}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{matrix} \psi_2 = \lambda \psi_1 \\ \psi_1 = \lambda \psi_2 \end{matrix}$$

$$\lambda = 0 \text{ or } \psi_1 = 0$$

$$\lambda = 0 \text{ or } \psi_2 = 0$$

$$\psi_2 = \frac{\partial \psi_1}{\partial t} + \sum_{n,j} \alpha_n \frac{\partial \psi_1}{\partial x}$$

$$\frac{\partial \psi_2}{\partial t} = (k^2 + \frac{\partial^2}{\partial x^2}) \psi_1 + \sum_{n,j} \alpha_n \left( \frac{\partial \psi_2}{\partial x} - \sum_{n,j} \alpha_n \frac{\partial \psi_1}{\partial x} \right)$$

$$\sum_j \psi_2 = \frac{\partial \psi_1}{\partial t} + \alpha \frac{\partial \psi_1}{\partial x} \quad \left( \frac{\partial \psi_2}{\partial x} - \alpha \frac{\partial \psi_1}{\partial x} \right)$$

$$\frac{\partial \psi_2}{\partial t} = \left( k^2 + \frac{\partial^2}{\partial x^2} \right) \psi_1 + \alpha \frac{\partial \psi_1}{\partial x \partial t}$$

$$= \alpha \frac{\partial \psi_2}{\partial x} + k^2 \psi_1 + (1 - \alpha^2) \frac{\partial \psi_1}{\partial x^2}$$

$$G = \begin{pmatrix} \alpha \frac{\partial}{\partial x} & 1 \\ k^2 + (1 - \alpha^2) \frac{\partial^2}{\partial x^2} & \alpha \end{pmatrix} \text{ Hermitian if } \alpha = \pm 1, k^2 = 1$$

Equation (8.3) gives

$$[k^2, [k, z]] = k[k, [k, z]] + [k, [k, z]]k = -(kz + zk)$$

Hence

$$\begin{aligned} [k^2, [k^2, z]] &= k[k^2, [k, z]] + [k^2, [k, z]]k = -(k^2z + 2kzk + zk^2) \\ &= -2(k^2z + zk^2) + (k^2z - 2kzk + zk^2) \\ &= -2(k^2z + zk^2) - k^2[k, [k, z]] \\ &= -2(k^2z + zk^2) + k^2z \end{aligned}$$

or

$$\begin{aligned} \frac{1}{2}[k^2, [k^2, z]] &= -(k^2 - \frac{1}{4}k^2)z - 2(k^2 - \frac{1}{4}k^2) \\ &= -k_1 k_2 z - 2k_1 k_2 \end{aligned} \quad (8.31)$$

where

$$k_1 = k + \frac{1}{2}h, \quad k_2 = k - \frac{1}{2}h.$$

(In general we shall take the suffix 1 attached to any action variable to denote the value of that variable increased by  $\frac{1}{2}h$ , and the suffix 2 to denote its value reduced by  $\frac{1}{2}h$ .)

From (8.23) and (8.26)

$$\begin{aligned} \frac{1}{2}[m^2, z] &= \frac{1}{2}[m_x^2 + m_y^2, z] = \frac{1}{2}(-m_x y - y m_x + m_y x + x m_y) \\ &= m_y x - m_x y + i\hbar z = m_y x - y m_x - x m_y - m_x y \end{aligned} \quad (8.32)$$

Similar relations hold for  $[m^2, x]$  and  $[m^2, y]$ . Hence

$$\begin{aligned} \frac{1}{2}[m^2, [m^2, z]] &= m_y[m^2, x] - m_x[m^2, y] + i\hbar[m^2, z] \\ &= 2m_y(y m_x - m_x y) - 2m_x(x m_y - m_y x) + i\hbar[m^2, z] \\ &= 2(m_x x + m_y y)m_z - 2(m_x^2 + m_y^2)z + i\hbar[m^2, z] \\ &= -2m^2 z + i\hbar[m^2, z] \\ &= -m^2 z - 2m^2. \end{aligned} \quad (8.33)$$

Comparing this with equation (8.31), we see that they agree if we take

$$m^2 = k_1 k_2 = k^2 - \frac{1}{4}h^2 \quad (8.34)$$

With  $k^2$  defined by (8.4), equation (8.31) follows from equation (8.33). We shall assume that equation (8.3) then follows from equation (8.31), although the present theory of square roots does not enable one to demonstrate this rigorously. Corresponding to (8.3) we have the equations

$$[k, [k, x]] = -x \quad [k, [k, y]] = -y \quad (8.5)$$

$$i\hbar \frac{\partial \psi}{\partial t} = G \psi \quad G = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2}(\frac{p^2}{m} + \frac{1}{2}) & 0 \end{pmatrix}$$

$$\psi_n = \psi_n(x, j)$$

$$x_{mn} = \int X_m x \psi_n dx$$

$$\dot{x}_{mn} = \int_j [\dot{X}_m x \psi_n + X_m x \dot{\psi}_n] dx$$

$$i\hbar \dot{x}_{mn} = \int_j \int_k [X_m x G_{nk} \psi_k - X_k G_{km} x \psi_n] dx = (xG - Gx)_{mn}$$

$$i\hbar \dot{x}(x'j', x''j'') = x' G(x'j', x''j'') - G(x'j', x''j'') x'' =$$

~~$G(x'1, x''1) = 0$~~

$$G(x'1, x''1) = 0$$

$$G(x'1, x''2) = \delta(x' - x'') \delta(y' - y'') \delta(z' - z'')$$

$$G(x'2, x''1) = -\frac{1}{2} \delta(x' - x'') \delta(y' - y'') \delta(z' - z'')$$

$$G(x'2, x''2) = 0$$

$$G(x'2, x''1) = -\frac{1}{2} \delta(x' - x'') \delta(y' - y'') \delta(z' - z'')$$

$$-\frac{1}{2} \delta(x' - x'') \delta(y' - y'') \delta(z' - z'')$$

$$-\frac{1}{2} \delta(x' - x'') \delta(y' - y'') \delta(z' - z'')$$

$$-\frac{1}{2} \delta(x' - x'') \delta(y' - y'') \delta(z' - z'')$$

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$$

Reversed correspondence between rows & columns

$$\text{Define } \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 b_3 + a_2 b_1 & a_1 b_4 + a_2 b_2 \\ a_3 b_3 + a_4 b_1 & a_3 b_4 + a_4 b_2 \end{pmatrix}$$

Addition same as usual

$$a(b+c) = ab + ac$$

second relation  
works as  
for multiplication

$$a1 = a \text{ where } 1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$1a = a \text{ where } 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$1b = b \text{ where } 1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^2 = \begin{pmatrix} a_1(a_2+a_3) & a_1^2+a_1a_4 \\ a_3^2+a_1a_4 & a_4(a_2+a_3) \end{pmatrix}$$

$$\text{Define } \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 + a_2 b_3 & a_1 b_2 + a_2 b_4 \\ a_3 b_3 + a_4 b_1 & a_3 b_4 + a_4 b_2 \end{pmatrix}$$

If  $a$  and  $b$  commute  
 $(ab)_{mn} = a_{mn} b_{mn}$

$a$  &  $b$  commute if

$$a_3 - a_4 - b_3 - b_4 = 0$$

Labels apply to the diagonals instead of rows & columns?

Work out canonical transformations

$$\begin{matrix} a_{1mn} b_1 & a_{1mn} b_2 \\ a_{1mn} b_3 & a_{1mn} b_4 \end{matrix}$$

One can take as another action variable the quantity  $m_z (= p_\phi \text{ say})$ , since from (8.23)

$$\left. \begin{aligned} [p, [p, x]] &= [p, y] = -x \\ [p, [p, y]] &= [p, x] = -y \end{aligned} \right\} \begin{matrix} 8.6 \\ (8.45) \end{matrix}$$

These equations show that  $x$  and  $y$  are periodic functions of  $\phi$ , the variable conjugate of  $p$ , of period  $2\pi$ , and that all the coefficients in their Fourier expansions vanish except those of  $e^{i\phi}$  and  $e^{-i\phi}$  terms, and also, since  $[p, z] = 0$ , all the coefficients in the Fourier expansion of  $z$  vanish except those of terms independent of  $\phi$ .

The ordinary selection rules for  $p$  follow from this.

Begin a new page

### 59. ~~The~~ Motion of a Particle in a Central Field: the Angle Variables.

We must now consider how  $\theta$  and  $\phi$  are to be defined. On the classical theory an angle variable  $w$  of this type is defined by  $e^{iw}$  being equal to the square root of the ratio of two quantities that are conjugate imaginaries, i.e. by a relation of the type

$$e^{iw} = \left( \frac{a+ib}{a-ib} \right)^{\frac{1}{2}} \quad \begin{matrix} 9.11 \\ (8.51) \end{matrix}$$

where  $a$  and  $b$  are real. This, of course, makes  $w$  real, since if one writes  $-i$  for  $i$  in (8.51) it remains true. 9.11

On the quantum theory, however, there are two corresponding ways in which one might define  $e^{iw}$ , namely

$$e^{iw} = \left\{ (a+ib)(a-ib) \right\}^{\frac{1}{2}}$$

and

$$e^{iw} = \left\{ \frac{1}{(a-ib)} (a+ib) \right\}^{\frac{1}{2}}$$

but neither of these makes  $w$  real. The correct quantum generalisation of (8.51) is the more symmetrical relation 9.11

$$e^{iw} (a-ib) e^{iw} = a+ib. \quad \begin{matrix} 9.12 \\ (8.52) \end{matrix}$$

This becomes, when one equates the conjugate imaginaries of both sides,

$$e^{-iw} (a+ib) e^{-iw} = a-ib,$$

which is equivalent to (8.52), so that  $w$  defined in this way is real. We may solve (8.52) for  $e^{iw}$  in either of two ways, i.e. 9.12

$$e^{iw} (a-ib) e^{iw} = (a+ib)(a-ib)$$

giving

$$e^{iw} (a-ib) = \left\{ (a+ib)(a-ib) \right\}^{\frac{1}{2}} = \left\{ (a+ib)(a-ib) \right\}^{-\frac{1}{2}} (a+ib)(a-ib)$$

so that

$$e^{iw} = \left\{ (a+ib)(a-ib) \right\}^{-\frac{1}{2}} (a+ib); \quad \begin{matrix} 9.13 \\ (8.53) \end{matrix}$$

or alternatively

$$(a-ib) e^{iw} (a-ib) e^{iw} = (a-ib)(a+ib)$$

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = k^2 \psi$$

$$-k^2 + E^2 = m^2 c^2$$

$$\psi = e^{i(kx + \mu t)}$$

$$v^2 - \mu^2 = k^2$$

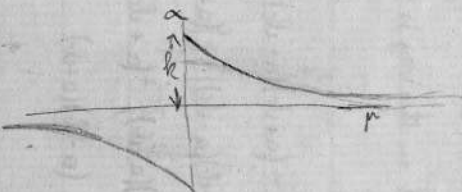
Initially  $\psi = \delta(x) = \int_{-\infty}^{\infty} e^{i\mu x} d\mu$

$$\psi = \int e^{i(kx + \mu t)} d\mu = \int e^{i\mu(x + vt)} \left(1 + i\mu ct + \frac{d^2 \psi}{dt^2} + \dots\right) d\mu$$

$$v = \pm \sqrt{k^2 + \mu^2} = \mu + \alpha$$

$$\alpha = \pm \sqrt{k^2 + \mu^2} - \mu = \mu \left[1 + \frac{k^2}{2\mu^2} - 1\right] \text{ } \mu \text{ large}$$

$\alpha$  always small



$$\psi \propto e^{-\mu |x|} e^{i\mu t}$$

$$\mu^2 = k^2 + \alpha^2$$

$$\text{for } |x| \ll |x|$$

$$\int_{-\infty}^{\infty} \left[ e^{i(\mu x + \sqrt{k^2 + \mu^2} t)} + e^{i(\mu x - \sqrt{k^2 + \mu^2} t)} \right] d\mu = 0$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2 + \mu^2}} \left[ e^{i(\mu x + \sqrt{k^2 + \mu^2} t)} - e^{i(\mu x - \sqrt{k^2 + \mu^2} t)} \right] d\mu = 0$$

$$\int_{-\infty}^{\infty} \left[ e^{i(\mu x + t \sqrt{k^2 + \mu^2})} \left(a + \frac{t}{\sqrt{k^2 + \mu^2}}\right) + e^{i(\mu x - t \sqrt{k^2 + \mu^2})} \left(a - \frac{t}{\sqrt{k^2 + \mu^2}}\right) \right] d\mu$$

$$\begin{aligned} \int_0^{\infty} e^{i(\mu x + \sqrt{k^2 + \mu^2} y)} d\mu &= \left[ \int_0^k + \int_k^{\infty} \right] e^{i(\mu x + \sqrt{k^2 + \mu^2} y)} d\mu \\ &= \int_k^{\infty} e^{i(\mu x + y \mu (1 + \frac{k^2}{2\mu^2} - \frac{k^4}{8\mu^4} - \dots))} d\mu + \int_0^k e^{i(\mu x + y k (1 + \frac{k^2}{2\mu^2} - \frac{k^4}{8\mu^4} - \dots))} d\mu \\ &= k \int_1^{\infty} e^{ik \left[ \frac{x}{k} \lambda + \frac{y}{k} \left( \frac{1}{\lambda} - \frac{1}{2} \frac{1}{\lambda^3} + \dots \right) \right]} d\lambda + k \int_0^1 e^{ik \left[ y + x \lambda + \frac{1}{2} y \lambda^2 - \frac{1}{8} y \lambda^4 + \dots \right]} d\lambda \end{aligned}$$

$$\mu = k\lambda$$

$$\mu = k\lambda$$

$$\int_0^{\infty} e^{ik(\lambda x + \sqrt{1 + \lambda^2} y)} d\lambda$$

$$x > y \text{ or } < -y$$

$$\text{near } k = ky, \text{ near } k = kx$$

$$y = x$$

$$\int e^{ik(a z + \sqrt{1 + z^2})} dz$$

$$\int_0^{\theta} e^{ik(a r e^{i\theta} + \sqrt{1 + r^2 e^{2i\theta}})} i r e^{i\theta} d\theta = i \int_0^{\theta} e^{i[k(a+1)r e^{i\theta} + \theta + \frac{1}{2} r^2 e^{-i\theta} + \dots]} d\theta$$

very large

$$= i r \int_0^{\theta} e^{i[k(a+1)r \cos \theta + \theta + \frac{1}{2} r^2 \cos \theta + \dots]} e^{-k(a+1)r \sin \theta + \frac{1}{2} r^2 \sin \theta + \dots} d\theta$$

Vanishes for  $a > 1, \theta > 0$

Should vanish when  $x > y$  or  $x < -y$   $a > 1$  or  $a < -1$

$$\int_0^{\alpha} f(\theta) e^{-R\theta} d\theta = \left[ f(\theta) \frac{e^{-R\theta}}{-R} \right]_0^{\alpha} - \int_0^{\alpha} f'(\theta) \frac{e^{-R\theta}}{-R} d\theta = \frac{f(0)}{R} + \frac{1}{R^2} \dots$$



## §12. The Elimination of the Nodes.

In the present § the work of §§ 8, 9 will be extended <sup>to</sup> the problem of the system with two electrons moving in <sup>an</sup> approximately central field of force. On the classical theory an initial simplification can be made, known as the elimination of the nodes, which consists in obtaining a contact transformation to a set of canonical variables, all of which except three are independent of the orientation of the system as a whole. It can be shown that the new variables may be taken to be the  $r, p_r, k$  and  $\theta$  of <sup>either</sup> each electron, the  $\theta$ 's now being measured from the line of nodes instead of <sup>from</sup> the line of intersection of the orbital plane with the plane  $xy$ , together with the resultant <sup>angular</sup> momentum  $j$ , with the azimuth  $\psi$  <sup>of the line of nodes</sup> about the direction of this resultant <sup>angular</sup> momentum for conjugate variable, and the component  $M_z$  of total angular momentum in a given direction,  $z$  say, with the azimuth of the direction of resultant angular momentum about the  $z$  axis for conjugate variable. All the variables except the last three are independent of the orientation of the system as a whole. The condition for this on the quantum theory must of course be expressed analytically, and is then that the variables are invariant under the linear transformation

$$\left. \begin{aligned} \bar{x} &= l_1 x + l_2 m_1 y + m_1 z & \bar{p}_x &= l_1 p_x + m_1 p_y + n_1 p_z \\ \bar{y} &= & \bar{p}_y &= \\ \bar{z} &= l_3 x + m_3 y + n_3 z & \bar{p}_z &= \end{aligned} \right\} (12.1)$$

where the coefficients  $l, m, n$  are c-numbers satisfying the conditions that they satisfy on the classical theory, namely

$$l_1^2 + m_1^2 + n_1^2 = 1, \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \quad \text{etc.}$$

On the quantum theory, we may still define <sup>the</sup>  $r, p_r, k$  <sup>of the first electron</sup> by (8.11) (8.12) and (8.4), as these variables are obviously invariant under the transformation (12.1). If we use dashed letters to refer to the second electron, we can define  $r', p_r', k'$  in a corresponding way, and then  $r', p_r'$  and  $k'$  will commute with  $r, p_r, k$ , so that these six quantities may be taken to be six of the new variables.

The components of total angular momentum are

$$M_x = m_x + m_x' \quad M_y = m_y + m_y' \quad M_z = m_z + m_z'$$

It is now easily verified that

$$[M_z, x] = y \quad [M_z, y] = -x \quad [M_z, z] = 0 \quad (12.23)$$

$$[M_z, z] = 0 \quad (12.24)$$

$$(\xi, \eta) = \int (\xi | \lambda) \lambda d\lambda(\eta)$$

$(\xi, \eta)$  is degenerate on the right when a  $(\eta | \cdot)$  exists such that

$$\int (\xi, \eta)(\eta | \cdot) d\eta = 0$$

$$a_{rn} = \sum_{\lambda} (r | \lambda) (\lambda | n)$$

$$b_{rn} = \sum_{\lambda} (r | \lambda) g(\lambda | n)$$

$$|a_{rn} + \gamma b_{rn}| = 0 \text{ when } f(\lambda) + \gamma g(\lambda) = 0$$

$$e^{Ht} \langle x | x' \rangle = \int \langle x | \xi' \rangle e^{f(\xi')t} \langle \xi' | x' \rangle d\xi'$$

H is diagonal in  $\xi$  scheme

$$f(\xi) \quad f_2(\xi)$$

1 is propagated with vel. c  
2 is -c

$$H(x', x'') = -\alpha \delta(x' - x'' - c\delta t)$$

$$\begin{matrix} 1 & 2 & = & -\alpha \delta(x' - x'' - c\delta t) \\ 2 & 1 & = & \alpha \delta(x' - x'' - c\delta t) \\ 1 & 1 & = & -\alpha \delta x' - x'' + 0 \end{matrix}$$

$$e^{H\delta t} \langle x' | x'' \rangle = \delta(x' - x'' + i c \delta t)$$

$$e^{H\delta t} \psi(x') = \psi(x' + i c \delta t) = \lambda \psi(x')$$

$$\psi(x) = e^{i p x} \quad \lambda = e^{-p c \delta t}$$

$$e^{Ht} \langle p' | p'' \rangle = \int e^{i(p' - p'')x''} \delta(x' - x'' + i c \delta t) dx''$$

$$= \int e^{i(p' - p'')x'' - p'' c \delta t} dx''$$

$$= e^{i p'' c \delta t} \delta(p' - p'')$$

$$H(p' | p'') = +p'' c \delta(p' - p'')$$

$$H(x' | x'') = \int p' c e^{i p' (x' - x'') + i p' c \delta t} dp' = c \delta(x' - x'')$$

$$= \int \langle x' | \xi_1 \rangle e^{f(\xi_1)t} \langle \xi_1 | x'' \rangle d\xi_1 + \int \langle x' | \xi_2 \rangle e^{f(\xi_2)t} \langle \xi_2 | x'' \rangle d\xi_2$$

$$\psi_1(x, t + \delta t) = e^{-(1 - \alpha \delta t)} \psi_1(x - c \delta t, t) + \alpha \delta t \psi_2(x + c \delta t, t)$$

$$e^{H\delta t} \langle x' | x'' \rangle = (1 - \alpha \delta t) \delta(x' - x'' - c \delta t) + \alpha \delta t$$

$$\begin{matrix} 1 & 2 & = & \alpha \delta t \delta(x' - x'' + c \delta t) \\ 2 & 1 & = & \alpha \delta t \delta(x' - x'' - c \delta t) \\ 1 & 1 & = & (1 - \alpha \delta t) \delta(x' - x'' - c \delta t) \end{matrix}$$

$$e^{-\alpha \delta t}$$

$$(\alpha \delta t)^2 (n-1) e^{-\alpha \delta t (n-2)}$$

$$(\alpha \delta t)^2 (n-2) e^{-\alpha \delta t (n-2)} + (\alpha \delta t)^4$$

derived from (1). To avoid having two symbols  $i$  and  $j$  both denoting roots of  $-1$  we shall take  $j=i$ , and must then modify the above rules to read:— from any equation one may obtain another equation by writing  $-i$  for  $i$  wherever it occurs and at the same time writing  $-h$  for  $h$ , or by reversing the orders of all factors and at the same time writing  $-h$  for  $h$ , or by applying the two previous operations together, which <sup>reduces</sup> ~~comes~~ to reversing the orders of all factors and writing  $-i$  for  $i$ . This third operation applied to any quantity gives what may be defined as the conjugate imaginary quantity. A quantity is defined as real if it is equal to its conjugate imaginary.

The remainder of this section will be devoted to some simple analytical rules which will be of use in the subsequent work. All the symbols denote  $q$ -numbers except  $h, i$  or except when otherwise stated.

When forming the reciprocal of a quantity composed of two or more factors one must reverse their order, i.e.

$$\frac{1}{(xy)} = \frac{1}{y} \cdot \frac{1}{x} \tag{4}$$

This equation may be verified by multiplying each side by  $xy$  either in front or behind.

To differentiate the reciprocal of a quantity  $x$  one must proceed as follows;—

$$0 = \frac{d}{dt} \left( \frac{1}{x} \cdot x \right) = \frac{d}{dt} \left( \frac{1}{x} \right) \cdot x + \frac{1}{x} \dot{x}.$$

Hence, dividing by  $x$  behind, one gets

$$\frac{d}{dt} \left( \frac{1}{x} \right) = - \frac{1}{x} \dot{x} \frac{1}{x}.$$

The binomial theorem is the same as in ordinary algebra. Also one defines  $e^x$  by the same power series in  $x$  as in ordinary algebra. The ordinary exponential law, however, is not valid, i.e.  $e^{x+y}$  is not in general equal to  $e^x e^y$  except when  $xy = yx$ , or, as we may say, when  $x$  commutes with  $y$ .

The Poisson bracket expression  $[x, y]$  may be put equal to its classical theory value when there is no ambiguity concerning the orders of factors of products in this value.

Thus we have

What right have you to say that  $\frac{d}{dt} 1 = 0$ ?  $\frac{d}{dt} [1, H] = 0$ ? If so say so. What does the product of two  $q$ -numbers mean? a term of  $q$ -numbers? Not yet defined? except for periodic spaces? Why? By hypothesis? What?

expansion for  $(1+x)^n$  when  $n$  is a  $q$ -number

it can easily be seen that if  $x$  and  $y$  are fun of the  $h$ 's and  $q$ 's,  $[x, y]$  is the same fun of the  $h$ 's and  $q$ 's as the classical theory value expresses

function.

$$\left\{ \sum_{xyz} \left( p_x - \frac{e}{c} A_x \right)^2 + m^2 c^2 \right\} \psi = 0$$

$-m^2 c^2$

$A_x$  means

$$\int A(xyz, x'y'z't') \psi dx'dy'dz'dt'$$

$$A_x = \frac{\frac{dx'_y}{ds}}{(x'_\mu - x_\mu) \frac{dx'_\mu}{ds}} \text{ for value of } x'_1 x'_2 x'_3 x'_4 \text{ which makes } (x_\mu - x'_\mu)^2 = 0$$

$$= \frac{p'_x - \frac{e}{c} A'_x}{(x'_\mu - x_\mu) (p'_x - \frac{e}{c} A'_x)}$$

$$\left( p_x - \frac{e}{c} A_x \right)^2 (x'_\mu - x_\mu) (p'_x - \frac{e}{c} A'_x) (x'_\sigma - x_\sigma) (p'_\sigma - \frac{e}{c} A'_\sigma) = \left\{ p_x (x'_\mu - x_\mu) (p'_x - \frac{e}{c} A'_x) - \frac{e}{c} (p'_x - \frac{e}{c} A'_x) \right\}^2$$

$$A_x A_x = \frac{1}{\left\{ (x'_\mu - x_\mu) \frac{dx'_\mu}{ds} \right\}^2} = \frac{1}{\left[ \frac{d}{ds} (x'_\mu - x_\mu) \right]^2}$$

$$A_x \left( p_x - \frac{e}{c} A_x \right)$$

$$\left\{ \left\{ p_x (x'_\mu - x_\mu) (p'_x - \frac{e}{c} A'_x) - \frac{e}{c} p'_x \right\}^2 + m^2 c^2 \left[ (x'_\mu - x_\mu) / p_x \right]^2 \right\} \psi = 0$$

$\psi$  very small except when  $(x'_\mu - x_\mu)^2$  is very small

$$\chi = \sum_n a_n e^{i \omega_n (t - x/c)} \chi_n$$

Klein's factor

$$\exp \left[ (x_\mu - x'_\mu)^2 \frac{m^2 c^2}{\hbar^2} \right]^{1/2} e^{i \phi}$$

$\psi = \psi_0 f(x)$



$$f(x) = \frac{e^{-x^2/a^2}}{a}$$

$$f'(x) = \frac{-2x}{a^2} e^{-x^2/a^2}$$

$$f''(x) = \frac{8x^2}{a^4} e^{-x^2/a^2} - \frac{2}{a^2} e^{-x^2/a^2}$$

$$f(x) = e^{-x^2/a^2}$$

$$f'(x) = \frac{-2x}{a^2} e^{-x^2/a^2}$$

$$f''(x) = \left( \frac{4x^2}{a^4} - \frac{2}{a^2} \right) e^{-x^2/a^2}$$

$$= \frac{4x^2 - 2a^2}{a^4} f(x)$$

$$\{H - W + iV(t-t')\} \psi = 0$$

$$\psi = \psi_1 \psi_2$$

$$\psi_2 (H - W) \psi_1 + i \psi_1 (t-t') V \psi_2 = 0$$

$$\frac{(H - W) \psi_1}{\psi_1} = -i(t-t') \frac{V \psi_2}{\psi_2} = \text{const.}$$

$$\{(H - W) V + i(t-t')\} \psi = 0$$

$$\psi = \psi_1 \psi_2$$

$$(H - W) \psi_1 V \psi_2 + i(t-t') \psi_1 \psi_2 = 0$$

$$\frac{(H - W) \psi_1}{\psi_1} = -i(t-t') \frac{V \psi_2}{V \psi_2} = \text{const.}$$

to the occurrence of the term  $k_1 k_2 / r^3 = (k^2 - \frac{1}{4} h^2) / r^3$  in (15) corresponding to the term  $k^2 / r^3$  in (12). There is thus no integral of (12) of the form (13).

We can, however, easily get an integral of (12) by making a small change in (13). We may transform from the variables  $r, \theta, p_r, k$  to the variables  $r, \theta', p_r, k'$ , where

$$k' = \sqrt{k^2 + \frac{1}{4} h^2}, \quad \theta' = \theta k' / k,$$

which are canonical since

$$[\theta', k'] = [\theta, k'] \frac{k'}{k} = \frac{k}{\sqrt{k^2 + \frac{1}{4} h^2}} \frac{k'}{k} = 1.$$

Now try

$$1/r = a_0 + a_1 e^{i\theta'} + a_2 e^{-i\theta'}. \quad (13')$$

Proceeding exactly as before, but using the Hamiltonian function

$$H = \frac{1}{2m} \left( p_r^2 + \frac{k'^2 - \frac{1}{4} h^2}{r^2} \right) - \frac{e^2}{r},$$

we find that

$$\frac{d}{dt} e^{i\theta'} = \frac{i}{m} e^{i\theta'} \frac{k'_1}{r^2}, \quad \frac{d}{dt} e^{-i\theta'} = -\frac{i}{m} e^{-i\theta'} \frac{k'_2}{r^2}$$

where

$$k'_1 = k' + \frac{1}{2} h, \quad k'_2 = k' - \frac{1}{2} h,$$

and further that

$$m \dot{p}_r = \frac{k'_1 k'_2}{r^3} - \frac{a_0 k'_1 k'_2}{r^2} = \frac{k^2}{r^3} - \frac{a_0 k^2}{r^2},$$

which agrees with (12) if we take  $a_0 = m e^2 / k^2$ . Hence (13') is an integral of the equation of motion (12).

This means that the orbit of the electron is an ellipse with a rotating apse line. If the Cartesian co-ordinates  $x = r \cos \theta$  and  $y = r \sin \theta$  are expanded in multiple Fourier series, two angle variables will be required, which will give two orbital frequencies. There must therefore necessarily be a two-fold infinity of energy levels, which disagrees with experiment (when one disregards the relativity fine-structure of the hydrogen spectrum). Hence one is forced to modify the Hamiltonian in order to make the motion degenerate. The Hamiltonian which makes the quantum orbit most closely resemble the classical orbit is

$$H = \frac{1}{2m} \left( p_r^2 + \frac{k_1 k_2}{r^2} \right) - \frac{e^2}{r}, \quad (16)$$

$$\bar{x} = x + \frac{\epsilon T}{c^2} \quad \bar{x}' = x' + \frac{\epsilon T}{c^2}$$

$$\bar{t} = T + \frac{\epsilon x}{c^2} \quad \bar{t}' = T' + \frac{\epsilon x'}{c^2}$$

$$\begin{aligned} \bar{T} &= T \\ \bar{x} &= x + \epsilon \bar{T} + \dot{\bar{x}} (\bar{T} - \bar{t}) = x + \epsilon \bar{T} - \dot{\bar{x}} \epsilon \frac{x}{c^2} \\ \bar{x}' &= x' + \epsilon \bar{T}' + \dot{\bar{x}}' (\bar{T}' - \bar{t}') \\ &= x' + \epsilon \bar{T}' - \dot{\bar{x}}' \epsilon \frac{x'}{c^2} \end{aligned}$$

$$\bar{x} = x - \frac{\partial H}{\partial p_x} \epsilon \frac{x}{c^2}$$

$$\bar{x}' = x' - \frac{\partial H}{\partial p_{x'}} \epsilon \frac{x'}{c^2}$$

$$\begin{aligned} [\bar{x}, \bar{x}'] &= [x, x'] - [x, \frac{\partial H}{\partial p_x}] \epsilon \frac{x'}{c^2} + [x', \frac{\partial H}{\partial p_{x'}}] \epsilon \frac{x}{c^2} \\ [x, x'] &= 0 \end{aligned}$$

$$H = p_1^2 + b p_2^2 b^{-1} + \cos(q_1 - b q_2 b^{-1})$$

$$\begin{aligned} [p_1 + b p_2 b^{-1}, q_1 - b q_2 b^{-1}] &= [b p_2 b^{-1}, q_1] - [p_1, b q_2 b^{-1}] \\ &= [b, q_1] p_2 b^{-1} - b p_2 b^{-1} [b, q_1] b^{-1} - [p_1, b] q_2 b^{-1} + b q_2 b^{-1} [p_1, b] b^{-1} \end{aligned}$$

$$\begin{aligned} p_1 + b p_2 b^{-1} &= p_1 + p_2 + i \hbar [b, p_2] b^{-1} \\ q_1 - b q_2 b^{-1} &= q_1 - q_2 - i \hbar [b, q_2] b^{-1} \end{aligned}$$

$$\begin{aligned} H &= p_1^2 + p_2^2 + p_2 \Delta t + \cos(q_1 - q_2 - q_2 \Delta t) \\ &= p_1^2 + p_2^2 + \cos(p_1 - q_2) + \{p_2 + \sin(q_1 - q_2) q_2\} \Delta t \end{aligned}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = -\sin(q_1 - q_2)$$

$$\dot{q}_2 = 1$$

$$H = f(b p_1 b^{-1}, b p_2 b^{-1}, b q_1 b^{-1}, q_2 - q_1)$$

$$\begin{aligned} H &= f(b p_1 b^{-1}, b p_2 b^{-1}) * f_0(b q_1, q_2) \\ &= b f(p_1, p_2) b^{-1} * f_0 \end{aligned}$$

$$\dot{q}_2 = \frac{\partial f_0}{\partial p_2}(p, q)$$

$$\frac{\partial f}{\partial p_2}(p_1 + p_2 \Delta t, q_1 + q_2 \Delta t, p_2, q_2)$$

$$\frac{\partial f}{\partial p_2}(p_1, p_2 \Delta t, q_1 + q_2 \Delta t, p_2, q_2)$$

$$= \frac{\partial f}{\partial p_2}(p_1, p_2, q_1, q_2) + \Delta t \left\{ \frac{\partial^2 f}{\partial p_2 \partial p_1} p_1 + \frac{\partial^2 f}{\partial p_2 \partial q_1} q_1 \right\}$$

$$H = f(p, q) + p_2 \Delta t + \dots$$

$$\begin{aligned} &f(p_1 + p_2 \Delta t, p_2 - p_2, q_1 + q_2 \Delta t, q_2 - q_2) \\ &= f(p, q) + \frac{\partial f}{\partial p_1} p_2 \Delta t + \frac{\partial f}{\partial q_1} q_2 \Delta t \end{aligned}$$

$$\alpha(1-\rho_1)J' = \beta A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}} = A^{\frac{1}{2}} J A^{-\frac{1}{2}}$$

$$[q_1, A^{\frac{1}{2}}] = \frac{1-\rho_1}{2A^{\frac{1}{2}}}$$

$$\frac{2H'}{C} = \frac{\beta}{\alpha(1-\rho_1)J'} = A^{-\frac{1}{2}} \beta A^{-\frac{1}{2}} = A^{\frac{1}{2}} \frac{2H}{C} A^{-\frac{1}{2}}$$

$$[w, A] = \alpha(1-\rho_1) + \frac{2A^{\frac{1}{2}}(1-\rho_1)\alpha \cos \theta}{\beta} = \frac{\alpha(1-\rho_1)}{\beta} B \quad [w, B] = 0$$

$$A^{\frac{1}{2}} w A^{-\frac{1}{2}} = w$$

$$[\alpha q_1, J'] = \beta \left( \frac{1}{2A^{\frac{1}{2}}} B^{-1} A^{\frac{1}{2}} + A^{\frac{1}{2}} B^{-1} \frac{1}{2A^{\frac{1}{2}}} \right) = \frac{\beta}{2A^{\frac{1}{2}}} \left( \frac{1}{B} A + A \frac{1}{B} \right) \frac{1}{2A^{\frac{1}{2}}} + 2J' [\alpha q_1, J'] \alpha(1-\rho_1) = \beta \{ A^{\frac{1}{2}} B^{-2} A^{\frac{1}{2}} + A^{\frac{1}{2}} B^{-1} A \} \\ = \frac{1}{2} \alpha(1-\rho_1) (A^{\frac{1}{2}} J J' A^{\frac{1}{2}})$$

$$X = \rho_3 x - \frac{\rho_2 \rho_3}{1-\rho_1} y + \frac{1-\rho_1 \rho_3^2}{1-\rho_1} z = \frac{1}{(1-\rho_1)} \{ \rho_3 (q_1 + c q_4 + \frac{1}{2} \rho_2 \rho_3) - \rho_2 \rho_3 q_2 + (1-\rho_1 \rho_3^2) q_3 \} = \frac{\rho_3}{1-\rho_1} (q_1 + c q_4) + q_3$$

$$Y = \rho_2 x + \frac{1-\rho_1 \rho_2^2}{1-\rho_1} y - \frac{\rho_2 \rho_3}{1-\rho_1} z = \frac{\rho_2}{1-\rho_1} (q_1 + c q_4) + q_2$$

$$[X, J'] \alpha(1-\rho_1) = A^{\frac{1}{2}} \left[ \rho_3, \frac{\alpha}{B} \right] A^{\frac{1}{2}} = 2A^{\frac{1}{2}} \left( \rho_3 \frac{1}{B} - \frac{1}{B} \rho_3 \frac{1}{B} \right) A^{\frac{1}{2}} = 2 \rho_3 A^{\frac{1}{2}} \frac{2A^{\frac{1}{2}} \cos \theta}{B^2} A^{\frac{1}{2}}$$

$$X = \frac{\rho_3}{1-\rho_1} (q_1 + c q_4) + q_3 \frac{\rho_2 \rho_3 H/c}{(1-\rho_1) H/c} + \cos \theta (H) = A^{\frac{1}{2}} \left\{ \frac{\rho_3}{1-\rho_1} (q_1 + c q_4) - q_3 \frac{\rho_2 + \rho_3 H/c}{(1-\rho_1) H/c} \right\} A^{-\frac{1}{2}} + \cos \theta (H')$$

$$Y = \frac{\rho_2}{1-\rho_1} (q_1 + c q_4) + q_2 \frac{\rho_2 + \rho_3 H/c}{(1-\rho_1) H/c} - \frac{A \sin \theta q_1}{\alpha(1-\rho_1) H/c} + \cos \theta (H)$$

$$(1-\rho_1)X = \rho_3 (A^{\frac{1}{2}} q_1 A^{-\frac{1}{2}} + c q_4) - A^{\frac{1}{2}} q_3 A^{-\frac{1}{2}} \left( \frac{\rho_2}{H} + \rho_3 \right) + \cos \theta (H') = \rho_3 c q_4 - A^{\frac{1}{2}} q_1 A^{-\frac{1}{2}} \frac{\rho_2}{H} + \cos \theta (H')$$

$$X = \frac{\rho_3}{1-\rho_1} c q_4 - q_1 \frac{\rho_2}{(1-\rho_1) H/c} + \cos \theta (H)$$

$$Y = \frac{\rho_2}{1-\rho_1} c q_4 - q_1 \frac{\rho_2}{(1-\rho_1) H/c} - \frac{A \sin \theta q_1}{\alpha(1-\rho_1) H/c}$$

$$\frac{2A^{\frac{1}{2}} \rho_2}{\beta(1-\rho_1)} \frac{2\alpha(1-\rho_1)J}{\beta}$$

$$\frac{2A^{\frac{1}{2}} \rho_2}{\alpha \beta} \sin w \frac{1}{(1-\rho_1)} \frac{2\alpha(1-\rho_1)J}{\beta} - A \sin w \frac{2\alpha(1-\rho_1)J}{\alpha \beta(1-\rho_1)}$$

$$= 2A \sin w \left( \frac{2 \rho_2^2 J}{\beta^2} - \frac{J}{\beta} \right)$$

$$\text{Integrating: } 4A^2 \left\{ \frac{\rho_2^2}{\beta^2} \frac{J(J+h)}{\beta^2} + \frac{(h^2 - \rho_2^2) J(J+h)}{\beta^2} \right\} = \frac{4A^2}{\beta^2} (\rho_2^2 h^2 + h^4 - h^2 \rho_2^2 + \rho_2^2 h^2) (J(J+h))$$

$$= \frac{A^2}{4 \sin^2 \theta} \{ \rho_2^2 \rho_3^2 + \rho_2^4 - 2\rho_2^2(1-\rho_1) + (1-\rho_1)^2 \} J(J+h)$$

$$\{ \} = (1-\rho_1) \{ (1-\rho_1) \rho_2^2 - 2\rho_2^2 + 1 - \rho_1^2 \} \\ = (1-\rho_1)^2 (1-\rho_2^2)$$

$$= \frac{A^2 (1-\rho_2^2)}{4 \sin^2 \theta \alpha^2} [1 + \gamma(1-\rho_1)]$$

$$\{ \} = \frac{1}{\rho_3} \{ (1-\rho_1)^2 - (1-\rho_1)(\rho_2^2 \rho_3^2) \} \\ = \frac{1}{\rho_3} (1-\rho_1) \{ 1 - \rho_1 - \rho_2^2 \rho_3^2 \} = -\frac{\rho_2}{\rho_3} (1-\rho_1)^2$$

$$(1-\rho_1 \rho_2^2) X + \rho_2 \rho_3 Y = \frac{(1-\rho_1 \rho_2^2)(1-\rho_1)}{\rho_3} X + \rho_2 \frac{(1-\rho_1)}{\rho_2} Y = (1-\rho_1 \rho_2^2 \rho_3^2) (1-\rho_1) X + \frac{(1-\rho_1 \rho_2^2)}{\rho_3} - \rho_2^2 \rho_3^2 Z : \rho_3 x - \rho_2 z$$

$$dnc^2 = \left(r_1 + \frac{e}{c} \phi\right)^2 - \left(r_1 + \frac{e}{c} A_1\right)^2 - \left(r_2 + \frac{e}{c} A_2\right)^2 - \left(r_2 + \frac{e}{c} A_2\right)^2 \quad \text{for charge } -e.$$

Pot. produced at \$(x,y,z)\$ by charge \$-e\$ at \$(x',y',z')\$ is if \$(x-x')^2 + (y-y')^2 + (z-z')^2 - c^2(t-t')^2 = 0\$

$$\phi = \frac{-ec}{c^2(t-t') - (x-x')^2 - (y-y')^2 - (z-z')^2} = \frac{-ec \frac{dt}{ds}}{c^2(t-t') \frac{dt}{ds} - (x-x') \frac{dx}{ds} - \dots}$$

$$A_x = \frac{-e \frac{dx}{ds}}{c^2(t-t') \frac{dt}{ds} - (x-x') \frac{dx}{ds} - \dots}$$

$$A_i = \frac{-e \frac{dx_i}{ds}}{g_{kl}(x_k - x'_k) \frac{dx_l}{ds}}$$

$$p'_i + \frac{e}{c} A_i = m \frac{dx'_i}{ds} - \frac{e^2 \frac{dx_i}{ds}}{g_{kl}(x_k - x'_k) \frac{dx_l}{ds}} = m \frac{dx'_i}{ds} \frac{dx_l}{ds} \frac{g_{kl}(x_k - x'_k)}{g_{kl}(x_k - x'_k) \frac{dx_l}{ds}} - \frac{e^2 \frac{dx_i}{ds}}{c \frac{dx_l}{ds}}$$

$$g_{ij} \left( m \frac{dx'_i}{ds} \frac{dx_l}{ds} \frac{g_{kl}(x_k - x'_k)}{g_{kl}(x_k - x'_k) \frac{dx_l}{ds}} - \frac{e^2 \frac{dx_i}{ds}}{c \frac{dx_l}{ds}} \right) \left( m \frac{dx'_j}{ds} \frac{dx_m}{ds} \frac{g_{mn}(x_n - x'_n)}{g_{mn}(x_n - x'_n) \frac{dx_m}{ds}} - \frac{e^2 \frac{dx_j}{ds}}{c \frac{dx_m}{ds}} \right) = m^2 c^2 \left\{ g_{ij} g_{kl} g_{mn} (x_k - x'_k) (x_n - x'_n) \frac{dx_l}{ds} \frac{dx_m}{ds} \right\}$$

$$\left\{ g_{kl} (x_k - x'_k) \frac{dx_l}{ds} \right\}^2 \left( m^2 c^2 - m^2 g_{ij} \frac{dx'_i}{ds} \frac{dx'_j}{ds} \right) = - \frac{2e^2 m^2}{c} g_{ij} \frac{dx_i}{ds} \frac{dx'_j}{ds} \frac{dx_m}{ds} g_{mn} (x_n - x'_n) + \frac{e^4}{c^2} g_{ij} \frac{dx_i}{ds} \frac{dx_j}{ds}$$

Can we find such a relation which commutes with \$(x-x')^2 + (y-y')^2 + (z-z')^2 - c^2(t-t')^2\$?

$$A_\mu = \frac{+e \frac{dx_\mu}{ds}}{(x_\mu - x'_\mu) \frac{dx_\mu}{ds}}$$

$$A = A(x) \quad B = B(x')$$

$$AB(\text{rel. int.}) = \frac{1}{2} \int (\psi_k \psi'_k - \psi'_k \psi_k) AB (x_k x'_k - x'_k x_k) \frac{dx_k}{ds} \frac{dx'_k}{ds}$$

$$= \frac{1}{2} \int \psi_k A \psi'_k \frac{dx_k}{ds} \int \psi'_k B \psi_k \frac{dx'_k}{ds} dt$$

$$+ \frac{1}{2} \int \psi'_k A \psi_k \frac{dx'_k}{ds} \int \psi_k B \psi'_k \frac{dx_k}{ds} dt$$

$$= \frac{1}{2} \int \psi_k A \psi_k \frac{dx_k}{ds} \int \psi'_k B \psi'_k \frac{dx'_k}{ds} dt$$

$$- \frac{1}{2} \int \psi'_k A \psi'_k \frac{dx'_k}{ds} \int \psi_k B \psi_k \frac{dx_k}{ds} dt$$

$$\frac{1}{2} (AB + BA)(\text{rel. int.}) = \frac{1}{2} \left\{ A(x_k) B(x'_k) + A(x'_k) B(x_k) - A(x'_k) B(x'_k) - A(x_k) B(x_k) \right\}$$

By using quantity to replace the value of that quantity and the symmetrical relation are

$$i\dot{N}_r = \sum_p N_{rp} N_p \frac{1}{N_0} e^{i(\phi_r - \phi_p)/\hbar} - \sum_p N_{pr} N_p \frac{1}{N_0} e^{i(\phi_p - \phi_r)/\hbar} \quad (1)$$

The eigenfunctions for the perturbed system satisfy the <sup>time</sup> <sup>space</sup>  $H$  equation

$$i\hbar \frac{\partial \psi}{\partial t} = (H_0 + A) \psi$$

where  $H_0 + A$  is an operator. If  $\psi = \sum_r a_r \psi_r$  where the  $\psi_r$ 's are the eigenfunctions for the unperturbed system, and the  $a_r$ 's are function of the time only, then  $|a_r|^2$  gives the probability of the system being in the  $r$ th state at any time. The theory applies directly to an assembly of  $N$  similar independent systems if we multiply each of these  $a_r$  by  $N^{1/2}$  so as to make  $\sum_r |a_r|^2 = N$ . We now have that  $|a_r|^2$  is the probable number of systems in the  $r$ th state. The equation that determines the rate of change of the  $a_r$ 's [see equation (25) of the all-time paper]

$$i\dot{a}_r = \sum_p V_{rp} a_p$$

where the  $V_{rp}$ 's are the elements of the matrix representing  $V$ .

(p2)

We can

It is convenient to transform to the canonical variables  $N_r, \phi_r$  by the contact transformation

$$a_r = N_r^{1/2} e^{i\phi_r/\hbar} \quad a_r^* = N_r^{1/2} e^{-i\phi_r/\hbar}$$

This transformation makes the  $N_r$  and  $\phi_r$  real,  $N_r$  being equal to  $a_r a_r^*$  and  $\phi_r$  being the phase of the eigenfunction that represents the probable number of systems in state  $r$ . The Hamiltonian  $F$ , now becomes.

$$F = \sum_{r,p} V_{rp} N_r^{1/2} N_p^{1/2} e^{i(\phi_r - \phi_p)/\hbar}$$

and the equation of motion that determine the rate at which transition <sup>occur</sup> take the canonical form

$$\dot{N}_r = -\frac{\partial F}{\partial \phi_r} \quad \dot{\phi}_r = \frac{\partial F}{\partial N_r}$$

The co-ordinates that are the canonical conjugates of the numbers of atoms in the different states are the phases of the eigenfunctions.

A slightly more convenient way of putting the equations of motion in the Hamiltonian form may be obtained with the help of the quantities  $b_r = a_r e^{-iW_r t/\hbar}$ ,  $b_r^* = a_r^* e^{iW_r t/\hbar}$ ,  $W_r$  being the energy of the  $r$ th state. We have  $|b_r|^2 = |a_r|^2$  the probable number of systems in the  $r$ th state. For  $b_r$  we find

$$i\hbar \dot{b}_r = W_r b_r + i\hbar \dot{a}_r e^{-iW_r t/\hbar} = W_r b_r + \sum_p V_{rp} b_p e^{i(W_p - W_r)t/\hbar}$$

This differs from equation (4) only in the notation, a single suffix  $r$  being used to denote a stationary state instead of a set of numerical values  $\mathbf{R}$  for the variable  $\mathbf{R}$ . Equation (4) or (4') can still be written when the Hamiltonian cannot be expressed as an algebraic function of a set of canonical variables, but can still be represented by a matrix  $H(\alpha, \alpha')$ .

We now take  $b_r$  and  $i\hbar \dot{b}_r$  to be canonical conjugates instead of  $a_r$  and  $i\hbar \dot{a}_r$ . The equations will take the Hamiltonian form with the Hamiltonian function

$$F = \sum_{r,p} b_r^* H_{rp} b_p$$

(5) (3a)

$$\frac{1}{r} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \psi = \frac{\partial^2}{\partial r^2} r^2(l)$$

$$r = \kappa r_1$$

$$r_2 r_1 \rightarrow \kappa_2 r_1$$

$$\frac{\partial}{\partial r} = \frac{\partial \kappa}{\partial r} \frac{\partial}{\partial \kappa} + \frac{\partial r_1}{\partial r} \frac{\partial}{\partial r_1} = \frac{1}{r_1} \frac{\partial}{\partial \kappa}$$

$$\sigma > \epsilon \quad \frac{1}{r_{at}} = \frac{1}{r_t} \sum_n \left( \frac{r_0}{r_t} \right)^n P_n(\mu_{ra})$$

$$H\psi = \sum_k \left( \frac{\partial^2}{\partial r_k^2} r_k^2 + \frac{1}{\sin \theta_k} \frac{\partial}{\partial \theta_k} \sin \theta_k \frac{\partial}{\partial \theta_k} + \frac{1}{\sin^2 \theta_k} \frac{\partial^2}{\partial \phi_k^2} \right) \psi$$

$$+ \sum_j \sum_i \frac{1}{r_j} \sum_n \left( \frac{r_k}{r_j} \right)^n P_n(\mu_{kj}) \psi$$

$$\psi = \sum_{m_1, m_2, \dots} c_{m_1, m_2, \dots} r_1^{m_1} r_2^{m_2} r_3^{m_3} \dots S_{n_1}(\theta_1, \phi_1) S_{n_2}(\theta_2, \phi_2) S_{n_3}(\theta_3, \phi_3) \dots$$

$$P_n(\mu_{ra}) P_{m_1, k_1}(\theta_1, \phi_1) P_{m_2, k_2}(\theta_2, \phi_2) =$$

$$r_n = \kappa_n r_{n-1}$$

$$\mu_n = \frac{1}{r_{n-1}} P_{n,n-1}$$

$$\text{Initial variables } r_1, r_2, r_3, \dots, r_n$$

$$h_1, h_2, h_3, \dots, h_n$$

$$\text{Final } r_1, r_2, r_3, \dots, r_n$$

$$P_1, P_2, P_3, \dots, P_n$$

$$r_1 \geq r_2 \geq r_3 \dots \geq r_n \geq 0$$

$$\kappa_n = \frac{r_n}{r_{n-1}} \quad (n > 1)$$

$$P_n = r_n \frac{1}{r_{n+1}}$$

$$\mu_{kj} = \cos \theta_k \cos \theta_j + \sin \theta_k \sin \theta_j \cos(\phi_j - \phi_k)$$



$$g(\xi' \eta') = (\xi' | \eta') g_{\xi' \eta'} = \frac{1}{\sqrt{2\pi\hbar}} e^{i\xi' \eta' / \hbar} g_{\xi' \eta'}$$

$$g(\eta' \xi') = \frac{1}{\sqrt{2\pi\hbar}} e^{-i\xi' \eta' / \hbar} g_{\eta' \xi'}$$

$$g_{\eta' \xi'} = \sqrt{2\pi\hbar} e^{i\xi' \eta' / \hbar} g(\eta' \xi')$$

$$= \sqrt{2\pi\hbar} e^{i\xi' \eta' / \hbar} \iint (\eta'' | \xi'') g(\xi'' \eta'') (\eta'' | \xi') d\xi'' d\eta''$$

$$= \frac{\sqrt{2\pi\hbar}}{\sqrt{2\pi\hbar}} e^{i\xi' \eta' / \hbar} \iint g(\xi'' \eta'') e^{-i(\xi'' \eta' + \xi' \eta'') / \hbar} d\xi'' d\eta''$$

$$= \frac{1}{2\pi\hbar} e^{i\xi' \eta' / \hbar} \iint e^{i\xi'' \eta'' / \hbar} g_{\xi'' \eta''} e^{-i(\xi'' \eta' + \xi' \eta'') / \hbar} d\xi'' d\eta''$$

$$= \frac{1}{2\pi\hbar} \iint g_{\xi'' \eta''} e^{i(\xi'' - \xi')(\eta'' - \eta') / \hbar} d\xi'' d\eta''$$

$$(PQ)_{\xi' \eta'} = \sqrt{2\pi\hbar} e^{-i\xi' \eta' / \hbar} (ab)(\xi' \eta')$$

$$= \sqrt{2\pi\hbar} e^{-i\xi' \eta' / \hbar} \iint a(\xi' \eta'') d\eta'' (\eta'' | \xi'') d\xi'' b(\xi'' \eta')$$

$$= (2\pi\hbar)^{-1} e^{-i\xi' \eta' / \hbar} \iint a_{\xi' \eta''} e^{i\xi' \eta'' / \hbar} e^{-i\xi'' \eta' / \hbar} e^{i\xi'' \eta'' / \hbar} b_{\xi'' \eta'} d\xi'' d\eta''$$

$$= \frac{1}{2\pi\hbar} \iint a_{\xi' \eta''} b_{\xi'' \eta'} e^{-i(\xi' - \xi'')(\eta' - \eta'') / \hbar} d\xi'' d\eta''$$

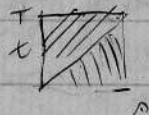
This is probably not true when the  $d\eta$  involve the time explicitly

$$- \frac{1}{\hbar^2} \sum_n c_n c_n^* \eta_{mn}^* \eta_{nm} \int_0^T K(t) e^{i(\omega_m - \omega_n)t/\hbar} dt \int_0^t K(s) e^{i(\omega_m - \omega_n)s/\hbar} ds$$

which reduces to

$$+ \frac{1}{\hbar^2} \sum_n \{k_n^2 - k_m^2\} |\eta_{mn}|^2 \left| \int_0^T K(t) e^{i(\omega_m - \omega_n)t/\hbar} dt \right|^2 \quad (29)$$

This gives the increase in the number of atoms in the  $m$ -th state due to the perturbation from the time  $t=0$  to the time  $T$ .





in some way in which the electron in a ionized stellar atmosphere  
settles into these quantized orbits of lowest energy when the gas is ionized its electrons

$$\left[ \left( p_z - \frac{e}{c} A_z \right)^2 \right] \psi \quad X \text{ is a fn of } x, y, z$$

$$\psi = \sum x_{nk} \psi_k$$

$$\square \phi = 4\pi \rho$$

$$t' = t - \frac{e_1 x + e_2 y + e_3 z}{c}$$

Choose the  $\alpha$ 's such that

$$[\alpha_1, t'] = [\alpha_2, t'] = [\alpha_3, t'] = 0$$

$$[\alpha_4, t'] = 1$$

$$\text{Put } \alpha_4 = \sqrt{\alpha_1 \alpha_2 \alpha_3}$$

$$X(\alpha) \text{ chosen such that } [X, \alpha_4] = 0$$

$$\int X(\alpha') e^{i \left[ f(\alpha) - \alpha_4 - f(\alpha') + \alpha_4' \right] t' / \hbar} d\alpha_4'$$

$X(\alpha')$  is a fn of  $\alpha_4$  and  $\alpha_4'$   
only through  $(\alpha_4 - \alpha_4')$

$X$  has certain components  $X(\alpha')$

$t'$

$$(X(t' - t') X)(\alpha') = \frac{\delta'(\alpha_4 - \alpha_4') \delta(\alpha_1 - \alpha_1') \delta(\alpha_2 - \alpha_2')}{\delta \alpha_3 - \alpha_3'}$$

$$= \int \left[ X(\alpha_1, \alpha_2, \alpha_3, \alpha_4) - X(\alpha_1, \alpha_2, \alpha_3, \alpha_4') \right] d\alpha_4$$

defined by its classical values

Contact transf. can be made only for one  $q$ -value to another with same characteristic values. Perhaps there is no unique product of two  $q$ -values, which fits in with idea that  $\alpha$  group is completely

$$a_{mm} b_{mn} = b_{mn} a_{mm} \\ b_{mn} = 0 \text{ unless } a_{mm} = a_{nn}$$

$$\psi_{mn} = \bar{\psi}_m \psi_n$$

$$\psi_m^* = \sum \psi_n a_{nm}$$

$$\bar{\psi}_m^* = \sum \bar{\psi}_n \bar{a}_{nm}$$

$$x \psi_m^* = \sum x_{km}^* \psi_k^* = \sum x_{km}^* a_{nk} \psi_n$$

$$= x \sum a_{km} \psi_k = \sum a_{km} x_{nk} \psi_n$$

$$\sum_k x_{nk} a_{km} = \sum_k a_{nk} x_{km}^*$$

$$x^* = a^{-1} x a$$

$$\sum_k a_{nk} \psi_{km}^* = \sum_k \psi_{nk} a_{km} = \sum_k \bar{\psi}_n \psi_k a_{km} = \bar{\psi}_n \psi_m^*$$

$$\sum_k a_{nk} \psi_{km}^* = \sum_k \psi_{nk} a_{km} = \bar{\psi}_n \sum_k \psi_k a_{km} = \bar{\psi}_n \psi_m^*$$

$$\sum_{k,n} a_{nk} \bar{a}_{ne} \psi_m^* = \sum_k \bar{a}_{ne} \bar{\psi}_n \psi_m^* = i \bar{\psi}_e^* \psi_m^*$$

$$\sum_k \int a_{nk} \psi_{km}^* d\tau = \int \delta_{nk} a_{km} = a_{nm}$$

$$= \int \psi_{km}(x_1, x_2) \psi_{nm}(x_1', x_2') dx_1 dx_2 \\ = \iint \psi_{km}(x_1, x_2) \psi_{nm}(x_1', x_2') dx_1 dx_2 \\ \delta(x_1 - x_1') \delta(x_2 - x_2')$$

$$\int P_{mm',nn'} d\tau' = A_{mm',nn'}$$

$$\sum_{k,k'} A_{mm',kk'} A'_{kk',nn'} = P_{mm',nn'}$$

$$\int P_{mm',nn'} d\tau' = A'_{mm',nn'}$$

$$= \int_{k,k'} P_{mm',kk'} d\tau' \cdot \int P_{kk',nn'} d\tau$$

$$P_{mm',nn'} = (\psi_m \psi_{m'}' - \psi_{m'}' \psi_m) (\psi_n \psi_n' - \psi_n' \psi_n)$$

$$x y z \neq x' y' z'$$

$$A_{mm',kk'} = \bar{\psi}_m \psi_k \delta_{m'k'} + \bar{\psi}_{m'} \psi_k \delta_{mk} - \bar{\psi}_m \psi_{k'} \delta_{m'k} - \bar{\psi}_{m'} \psi_{k'} \delta_{mk}$$

$$A'_{kk',nn'} = \bar{\psi}_k' \psi_n' \delta_{k'n} + \bar{\psi}_{k'}' \psi_n' \delta_{kn} - \bar{\psi}_k' \psi_n \delta_{k'n} - \bar{\psi}_{k'}' \psi_n \delta_{kn}$$

$$\sum_{k,k'} A_{mm',kk'} A'_{kk',nn'} = 2 \bar{\psi}_m \psi_n \bar{\psi}_{m'}' \psi_n' + \bar{\psi}_m \psi_n \bar{\psi}_{m'} \psi_n \delta_{nn'} - \bar{\psi}_m \psi_{n'} \bar{\psi}_{m'} \psi_n - 2 \bar{\psi}_m \psi_n \bar{\psi}_{m'}' \psi_n' \\ + 2 \bar{\psi}_m \psi_n \bar{\psi}_{m'} \psi_n' - 2 \bar{\psi}_m \psi_n \bar{\psi}_{m'} \psi_n' - \bar{\psi}_m \psi_n \bar{\psi}_{m'}' \psi_n' \\ + \bar{\psi}_m \psi_n \bar{\psi}_{m'}' \psi_n' - \bar{\psi}_m \psi_n \bar{\psi}_{m'} \psi_n' - \bar{\psi}_m \psi_n \bar{\psi}_{m'}' \psi_n'$$

$$= 2 (\bar{\psi}_m \bar{\psi}_{m'}' - \bar{\psi}_{m'} \bar{\psi}_m') (\psi_n \psi_n' - \psi_n' \psi_n)$$

$$= \psi_{nn'} \psi_{mm'}$$

$$\sum_k a_{mk} b_{kn} = 0$$

$$x_{mn} = \sum_p x_{mk} a_{kn}$$

Matrix of rank  $p$  is multiplied by matrix of rank  $q$

Product matrix has a rank not greater than  $\min(p, q)$  and not less than  $n - p - q$   
 $n = \text{no. of rows or columns}$



$$\psi(\alpha, \alpha_2, \beta_1, \beta_2) \text{ orthogonal}$$

$$\psi(\alpha, \beta_2, \beta_1, \beta_2) = \theta_k(\alpha, \beta_1) \psi(\beta_2)$$

$$\psi_{\alpha\alpha'} = \sum_{\beta} \theta_{\alpha}(\alpha) \theta_{\beta}(\alpha) \theta_{\beta}(\alpha') \theta_{\alpha'}(\alpha')$$

$$\int \theta_{\alpha}(\alpha) \theta_{\beta}(\alpha) d\alpha = \delta_{\alpha\beta}$$

$$\int \theta_{\alpha}(\alpha) \theta_{\alpha'}(\alpha') d\alpha = \delta_{\alpha\alpha'}$$

$$\int \int \theta_{\alpha}(\alpha') \theta_{\alpha}(\alpha) \theta_{\beta}(\alpha) d\alpha d\alpha' = \int \theta_{\alpha}(\alpha') \delta_{\alpha\beta} d\alpha = \delta_{\alpha\beta} \theta_{\alpha}(\alpha')$$

$$\rho(\alpha, \alpha_2, \beta_1, \beta_2) = \psi(\alpha, \beta_1) \psi(\alpha_2, \beta_2)$$

$$\int \alpha(\alpha, \alpha_2) \rho(\alpha, \alpha_2, \beta_1, \beta_2) d\alpha d\alpha_2 = \alpha(\beta_1, \beta_2)$$

$$\rho(\alpha\alpha', \beta\beta') = \psi(\alpha\beta) \psi(\alpha'\beta')$$

$$\delta(x) = 0 \text{ when } x \neq 0$$

$$\int \delta(x) dx = 1$$

$$\int \rho(\alpha\alpha', \beta\beta) d\beta = \delta(\alpha - \alpha')$$

$$1(\alpha\alpha') = \delta(\alpha - \alpha')$$

$$\int \rho(\alpha\alpha', \beta\beta') d\alpha = \delta(\beta - \beta')$$

$$\int \rho(\alpha\alpha', \beta\beta') \alpha(\beta\beta') d\beta d\beta' = \alpha(\alpha\alpha')$$

$$\int \alpha(\alpha') y(\alpha'') d\alpha' = \int \rho(\alpha', \beta\beta') \alpha(\beta\beta') \rho(\alpha'', \beta''\beta''') y(\beta''\beta''') d\beta d\beta' d\beta'' d\beta''' d\alpha'$$

$$= \int \rho(\alpha'', \beta\beta'') \alpha(\beta\beta'') y(\beta'\beta'') d\beta d\beta' d\beta''$$

$$\int \int \rho(\alpha\alpha', \beta\beta') \rho(\alpha\alpha'', \beta''\beta''') y(\beta''\beta''') d\beta'' d\beta''' d\alpha' = \int \rho(\alpha'', \beta\beta'') y(\beta'\beta'') d\beta''$$

$$= \int \rho(\alpha'', \beta\beta'') \delta(\beta' - \beta'') y(\beta''\beta''') d\beta'' d\beta'''$$

$$\int \rho(\alpha\alpha', \beta\beta') \rho(\alpha\alpha'', \beta''\beta''') d\alpha' = \rho(\alpha\alpha'', \beta\beta''') \delta(\beta' - \beta'')$$

$$\rho = \sum_k \phi_k$$

$$\rho(\alpha\alpha', \beta\beta') = \sum_k \phi_k(\alpha\beta) \psi_k(\alpha'\beta')$$

$$\sum_{k\ell} \int \phi_k(\alpha\beta) \psi_k(\alpha'\beta') \phi_{\ell}(\alpha'\beta'') \psi_{\ell}(\alpha''\beta''') d\alpha' = \sum_k \phi_k(\alpha\beta) \psi_k(\alpha''\beta''') \delta(\beta' - \beta'')$$

$$= \sum_{k\ell} \phi_k(\alpha\beta) \psi_{\ell}(\alpha''\beta''') \delta_{k\ell} \delta(\beta' - \beta'')$$

$$\int \psi_k(\alpha'\beta') \phi_{\ell}(\alpha'\beta'') d\alpha' = \delta_{k\ell} \delta(\beta' - \beta'')$$

$$\int \phi_m(\alpha\beta') \psi_k(\alpha'\beta') \phi_{\ell}(\alpha'\beta'') d\alpha' d\beta' = \delta_{k\ell} \int \phi_m(\alpha\beta') \delta(\beta' - \beta'') d\beta' = \delta_{k\ell} \phi_m(\alpha, \beta'')$$

$$= \delta_{k\ell} \int \phi_m(\alpha'\beta'') \delta(\alpha - \alpha') d\alpha'$$

If  $\phi(x)$  is any of the improper functions we have to deal with  $\int_{-\infty}^{\infty} \phi(x) f(x) dx$  is always a continuous <sup>regular</sup> f of  $f$  for any regular  $f(x)$  and is one of our improper fns if  $f(x)$  is an improper f.

$$\eta_1(z'z') = -i\hbar \delta'(z_1' - z_1'') \delta(z_2'' - z_1')$$

$$\eta_1(\alpha'z') = \int (\alpha'z') d\alpha'. \quad \eta_1(z'z')$$

$(z_1'z_1')$  and  $(\alpha_2'z_1')$  are conj. imaginaries

$$\int (\alpha_2'z_1') \delta'(z_1' - z_1'') = (\alpha_2'z_1'')$$

$$\text{and } \int (\alpha_2''z_1') d\alpha_2'(z_1' - z_1'') = (\alpha_2''z_1'')$$

are conj. imaginaries

$$\eta_r + \frac{\partial \eta}{\partial z_r}$$

$$\begin{aligned} \delta'(z_1' - z_1'') \frac{\partial \eta}{\partial z_1''} &= \delta'(z_1' - z_1'') \\ &= \delta'(z_1' - z_1'') + \delta'(z_1' - z_1'') \frac{\partial \eta}{\partial z_1''} \end{aligned}$$

$$x\delta(x) = \delta(x)$$

$$\left( \frac{z_1' - z_1''}{z_1'' - z_1'''} - \frac{z_1''}{z_1' - z_1'''} \right) \delta(z_1'' - z_1''') \delta(z_1' - z_1''') d\alpha_1'''$$

$$= \frac{z_1'^2 - z_1'z_1'' - z_1''z_1''' + z_1'''^2}{(z_1'' - z_1''')(z_1' - z_1''')} = 1 + \frac{z_1'(z_1' - z_1'')}{(z_1'' - z_1''')(z_1' - z_1''')}$$

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx$$

$$f'(x) = -\frac{1}{x} f(x)$$

$$\frac{df}{dx} = -\frac{f}{x}$$

$$\log f = \log x + \text{const.}$$

$$f = \frac{C}{x}$$

$$g(\xi\xi'') = \int a(\xi\xi''\alpha') g(\alpha') d\alpha' d\alpha''$$

$$\xi' \xi'' = m, n.$$

$$\alpha' \alpha'' = q$$

$$fg(\xi\xi'') = \int a(\xi\xi''\alpha') d\alpha' d\alpha'' f(\alpha') d\alpha''' g(\alpha''\alpha''')$$

$$= \int f(\xi\xi''') d\xi''' g(\xi''\xi''') = \int a(\xi\xi'''\alpha') f(\alpha') d\alpha' d\alpha'' d\xi''' a(\xi''\xi'''\alpha''\alpha''') g(\alpha''\alpha''') d\alpha'' d\alpha'''$$

$$[a(\xi\xi'''\alpha') \delta(\alpha'' - \alpha''') = \int a(\xi\xi'''\alpha') d\xi''' a(\xi'''\xi''\alpha''\alpha''').]$$

$$\text{Suppose } g(\alpha'\alpha'') = g(\alpha') \delta(\alpha' - \alpha'') \quad f(\alpha'\alpha'') = f(\alpha') \delta(\alpha' - \alpha'').$$

$$[a_{mn}(q'q'') \delta(q' - q'') = \sum_k a_{mk}(q') a_{kn}(q'').]$$

$$\int a(\xi\xi''\alpha') f(\alpha') g(\alpha') d\alpha' = \int a(\xi\xi''\alpha') f(\alpha') d\alpha' d\xi''' a(\xi''\xi'''\alpha''') g(\alpha''') d\alpha''.$$

$$a(\xi\xi''\alpha') \delta(\alpha' - \alpha'') = \int a(\xi\xi'''\alpha') d\xi''' a(\xi'''\xi''\alpha''\alpha''')$$

$$a_{mn}(q') \delta(q' - q'') = \sum_k a_{mk}(q') a_{kn}(q'')$$

$$a(\xi\xi''\alpha')$$

$$a_{mk}(q') = \frac{\partial \psi_m}{\partial t} \bar{\psi}_k - \frac{\partial \bar{\psi}_k}{\partial t} \psi_m + \gamma \bar{\psi}_k \psi_m$$

$$a_{kn}(q'')$$

$$\sum_k a_{mk}(q') a_{kn}(q'') = \sum_k \left( \frac{\partial \psi_m}{\partial t} \bar{\psi}_k - \frac{\partial \bar{\psi}_k}{\partial t} \psi_m + \gamma \bar{\psi}_k \psi_m \right) \left( \frac{\partial \psi'_n}{\partial t} \bar{\psi}'_k - \frac{\partial \bar{\psi}'_k}{\partial t} \psi'_n + \gamma \bar{\psi}'_k \psi'_n \right)$$

$$= \gamma^2 \bar{\psi}_k \psi_m \delta(q' - q'') + \gamma \left( \psi_m \frac{\partial \bar{\psi}_k}{\partial t} - \frac{\partial \psi_m}{\partial t} \bar{\psi}_k \right) \delta(q' - q'') - \frac{\partial \psi_m}{\partial t} \frac{\partial \psi'_n}{\partial t} \delta(q' - q'')$$

$$+ \frac{\partial \psi_m}{\partial t} \bar{\psi}'_k \sum_n \bar{\psi}_k \frac{\partial \psi'_n}{\partial t} + \gamma \psi_m \bar{\psi}'_k \sum_n \bar{\psi}_k \frac{\partial \psi'_n}{\partial t} - \gamma \psi_m \bar{\psi}'_k \sum_n \frac{\partial \bar{\psi}_k}{\partial t} \psi'_n \frac{\partial \bar{\psi}'_k}{\partial t} \frac{\partial \psi'_n}{\partial t} \psi_m \bar{\psi}'_k$$

$$+ \frac{\partial \bar{\psi}'_k}{\partial t} \psi_m \sum_n \bar{\psi}_k \psi'_n \frac{\partial \bar{\psi}_k}{\partial t}$$

$$a_{mn}(q'q'') \delta(q' - q'') = \sum_k a_{mk}(q'q'') a_{kn}(q''q''')$$

$$\text{Put } a_{mn}(q'q'') = \sum_\lambda f_{m\lambda}(q') g_{\lambda n}(q'')$$

$$\sum_\lambda f_{m\lambda}(q') g_{\lambda n}(q'') \delta(q' - q'') = \sum_\lambda f_{m\lambda}(q') \sum_{\lambda'} \sum_k g_{k\lambda'}(q'') f_{k\lambda'}(q''') g_{\lambda' n}(q''').$$

$$= \sum_{\lambda, \lambda'} f_{m\lambda}(q') g_{\lambda' n}(q''') \cdot \sum_k g_{k\lambda'}(q'') f_{k\lambda'}(q''').$$

$$\int a_{mn}(q'q'') dq' = b_{mn}(q'')$$

$$b_{mn}(q''') = \sum_k b_{mk}(q'') b_{kn}(q''')$$

$$a_m^{(r)} = \delta_{m0} + \frac{1}{i\hbar c} \dot{\eta}_{m0} \int_0^T \kappa(t) e^{i\omega_{m0}t} dt + \frac{1}{i\hbar c^2} \sum_n \dot{\eta}_{mn} \dot{\eta}_{n0} \int_0^T \kappa(t) e^{i\omega_{mn}t} dt \int_0^T \kappa(s) e^{i\omega_{n0}s} ds$$

$$\dot{a}_m^{(r)} = a_m = c_m + c_m' + c_m'' + c_m''' + \dots$$

$$i\hbar c \dot{c}_m^{(r)} = \sum_n c_n^{(r-1)} \kappa \dot{\eta}_{mn} e^{i\omega_{mn}t} \quad \left| \quad c_m^{(r)}(t) = \frac{1}{i\hbar c} \sum_n \dot{\eta}_{mn} \int_0^T \kappa(\tau) e^{i\omega_{mn}\tau} d\tau c_n^{(r-1)}(t) \right.$$

$$c_m'''(T) = \frac{1}{(i\hbar c)^3} \sum_{n, k, j} \dot{\eta}_{mn} \dot{\eta}_{nk} \dot{\eta}_{j0} \int_0^T \kappa(\tau) e^{i\omega_{mn}\tau} d\tau \int_0^T \kappa(t) e^{i\omega_{nk}t} dt \int_0^T \kappa(s) e^{i\omega_{j0}s} ds.$$

$$\text{if } \ell \neq 1 \quad c_m = 0 \quad (m \neq 1)$$

$$\text{at time } T \quad c_m' = \frac{c_0}{i\hbar c} \dot{\eta}_{m0} \int_0^T \kappa(t) e^{i\omega_{m0}t} dt$$

$$c_m'' = \frac{c_0}{(i\hbar c)^2} \sum_n \dot{\eta}_{mn} \dot{\eta}_{n0} \int_0^T \kappa(\tau) e^{i\omega_{mn}\tau} d\tau \int_0^T \kappa(t) e^{i\omega_{n0}t} dt$$

$$c_m''' = \frac{c_0}{(i\hbar c)^3} \sum_{n, k, j} \dot{\eta}_{mn} \dot{\eta}_{nk} \dot{\eta}_{j0} \int_0^T \kappa(\tau) e^{i\omega_{mn}\tau} d\tau \int_0^T \kappa(t) e^{i\omega_{nk}t} dt \int_0^T \kappa(s) e^{i\omega_{j0}s} ds.$$

$$\bar{c}_m' c_m'' = \frac{|c_0|^2}{i\hbar^3 c^3} \sum_n \dot{\eta}_{mn} \dot{\eta}_{n0} \dot{\eta}_{0m} \int_0^T \kappa(\tau) e^{-i\omega_{m0}\tau} d\tau \cdot \int_0^T \kappa(t) e^{i\omega_{mn}t} dt \int_0^T \kappa(s) e^{i\omega_{n0}s} ds.$$



$$(Xt' - t'X)(\alpha'') = \int [X(\alpha'') \delta(\alpha'_1 - \alpha_1'') \delta(\alpha'_2 - \alpha_2'') \delta(\alpha'_3 - \alpha_3'') \delta(\alpha'_4 - \alpha_4'') \\ - \delta(\alpha_1 - \alpha_1') \delta(\alpha_2 - \alpha_2') \delta(\alpha_3 - \alpha_3') \delta(\alpha_4 - \alpha_4') X(\alpha' \alpha'')] d\alpha'_1 d\alpha'_2 d\alpha'_3 d\alpha'_4$$

$$= \int [X(\alpha, \alpha''_1 \alpha''_2 \alpha''_3 \alpha'_4) \delta(\alpha'_4 - \alpha_4'') - X(\alpha_1 \alpha_2 \alpha_3 \alpha'_4, \alpha'') \delta(\alpha_4 - \alpha_4')] d\alpha'_4$$

$$= \int \left[ \frac{\partial}{\partial \alpha'_4} X(\alpha, \alpha''_1 \alpha''_2 \alpha''_3 \alpha'_4) \delta(\alpha'_4 - \alpha_4'') + \frac{\partial}{\partial \alpha_4} X(\alpha_1 \alpha_2 \alpha_3 \alpha'_4, \alpha'') \delta(\alpha_4 - \alpha_4'') \right] d\alpha'_4$$

$$= \left[ \frac{\partial}{\partial \alpha'_4} X(\alpha, \alpha''_1 \alpha''_2 \alpha''_3 \alpha'_4) \right]_{\alpha'_4 = \alpha_4''} + \left[ \frac{\partial}{\partial \alpha_4} X(\alpha_1 \alpha_2 \alpha_3 \alpha'_4, \alpha'') \right]_{\alpha_4 = \alpha_4''}$$

$$= - \frac{\partial}{\partial \alpha_4''} X(\alpha'') - \frac{\partial}{\partial \alpha_4} X(\alpha'')$$

If this = 0 then  $X(\alpha'')$  involves  $\alpha_4$  and  $\alpha_4''$   
only through their difference.

Schubert

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$$-\frac{\hbar^2 \nabla^2}{2m} + \frac{1}{r \sin \theta}$$

$$dS = r^2 \sin \theta d\theta d\phi$$

$$\int \left( \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial \theta} r d\theta + \frac{\partial V}{\partial \phi} r \sin \theta d\phi \right)$$

$$= \int \left\{ \frac{\partial}{\partial r} \left( \frac{\partial V}{\partial r} r^2 \sin \theta \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial V}{\partial \theta} r \sin \theta \right) + \frac{\partial}{\partial \phi} \left( \frac{\partial V}{\partial \phi} r \sin \theta \right) \right\} dr d\theta d\phi$$

$$\nabla^2 V = \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} \left( \frac{\partial V}{\partial r} r^2 \sin \theta \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial V}{\partial \theta} r \sin \theta \right) + \frac{\partial}{\partial \phi} \left( \frac{\partial V}{\partial \phi} r \sin \theta \right) \right\}$$

$$= \frac{\partial^2 V}{\partial r^2} + \frac{2}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$$

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r} \frac{\partial V}{\partial r} = \frac{1}{r^2} \nabla^2 V$$

$$\frac{1}{r^2} \nabla^2 (rV) = \frac{1}{r^2} \nabla^2 (rV)$$

$$rV = e^{ikr}$$

$$\left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) V = \frac{1}{r^2} \nabla^2 V$$

$$V = r^m$$

$$m(m-1) + 2m = k(k+1)$$

$$V = e^S$$

$$\frac{\partial V}{\partial r} = e^S \frac{\partial S}{\partial r}$$

$$\frac{\partial^2 V}{\partial r^2} = e^S \frac{\partial^2 S}{\partial r^2} + e^S \left( \frac{\partial S}{\partial r} \right)^2$$

$$\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 S}{\partial \phi^2} + k(k+1)S = 0$$

$$\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( f' \sin \theta \right) - \frac{m^2}{r^2 \sin^2 \theta} f = k(k+1)f$$

$$f(\theta) = g(\sin \theta) = g(x)$$

$$g''(1-x^2) - 2g'x - \frac{m^2}{1-x^2} g = k(k+1)g$$

$$f(\theta) = g(\sin \theta)$$

$$\frac{df}{d\theta} = g' \cdot \sin \theta$$

$$f' \sin \theta = -g' \sin^2 \theta$$

$$\frac{\partial}{\partial \theta} (f' \sin \theta) = -g'' \sin^3 \theta - g' \cdot 2 \sin \theta \cos \theta$$

$$S = f(\theta) e^{ikr}$$

$$z_1(x_2 y z + y z x_2 + z x_2 y) - x_1(y x_2 z + x_2 z y + z y x_2)$$

$$(x_1 y z + y z x_1 + z x_1 y) x_2 - (y x_1 z + x_1 z y + z y x_1) x_2$$

$$(x_1 x_2 y z + y z x_1 x_2 + z x_1 x_2 y) - (x_1 x_2 z y + z y x_1 x_2 + y x_1 x_2 z) = \cancel{(x_1 x_2 [y z])}$$

theory (e.g. the question; - What is the fraction of the

$F_{x,y}$



$D \sin \alpha = a \sin \alpha$   
 $D \cos \alpha = -a \sin \alpha$   
 $D e^{i\alpha} = i e^{i\alpha}$   
 $D e^{-i\alpha} = -i e^{-i\alpha}$

Discuss general transf. to variables that satisfy eqns. of motion for a particular Hamiltonian. This transf. they will probably fit in better with the Hamiltonian equations

$$F(q, p, t) = 0 \quad W = -p_0 \quad t = q_0$$

$$F(q, p) = 0$$

$$X F = 0$$

$$\frac{dq_r}{ds} = \frac{\partial(XF)}{\partial p_r} = \frac{\partial X}{\partial p_r} F + X \frac{\partial F}{\partial p_r}$$

$$\bar{p}_r = p_r + \theta_r F$$

$$\bar{q}_r = q_r + \theta'_r F$$

$$F(\bar{p}, \bar{q}) = \bar{F}(\bar{p}, \bar{q})$$

$$\frac{\partial \bar{F}}{\partial q_r} = \frac{\partial F}{\partial q_0} \frac{\partial q_0}{\partial q_r} + \frac{\partial F}{\partial p_0} \frac{\partial p_0}{\partial q_r} = \frac{\partial F}{\partial q_0} (\delta_{r0} - \theta'_r \frac{\partial \bar{F}}{\partial q_r}) + \frac{\partial F}{\partial p_0} \theta_r \frac{\partial \bar{F}}{\partial q_r}$$

$$\frac{\partial \bar{F}}{\partial p_r} = \frac{\partial F}{\partial p_r} + \theta'_r \frac{\partial F}{\partial q_0} \frac{\partial \bar{F}}{\partial q_r} + \theta_r \frac{\partial F}{\partial p_0} \frac{\partial \bar{F}}{\partial q_r} = \frac{\partial \bar{F}}{\partial p_r} (1 + \theta'_r \frac{\partial \bar{F}}{\partial q_0} + \theta_r \frac{\partial \bar{F}}{\partial p_0})$$

$$\frac{\partial \bar{p}_r}{\partial t} \frac{\partial \bar{q}_s}{\partial t} - \frac{\partial \bar{p}_s}{\partial t} \frac{\partial \bar{q}_r}{\partial t} = (\delta_{rs} + \theta_r \frac{\partial F}{\partial p_t}) (\delta_{st} + \theta'_s \frac{\partial F}{\partial q_t}) - \theta_s \frac{\partial F}{\partial p_t} \theta'_r \frac{\partial F}{\partial q_t} = \delta_{rs} + \theta'_r \frac{\partial F}{\partial q_s} + \theta_r \frac{\partial F}{\partial p_s}$$

$$\frac{\partial x}{\partial q_r} \frac{\partial y}{\partial p_r} = \left\{ \frac{\partial x}{\partial q_0} (\delta_{r0} + \theta'_r \frac{\partial F}{\partial q_r}) + \frac{\partial x}{\partial p_0} \theta_r \frac{\partial F}{\partial q_r} \right\} \left\{ \frac{\partial y}{\partial p_t} (\delta_{rt} + \theta_t \frac{\partial F}{\partial p_r}) + \frac{\partial y}{\partial q_t} \theta'_t \frac{\partial F}{\partial p_r} \right\}$$

$$= \frac{\partial x}{\partial q_r} \frac{\partial y}{\partial p_r} + \frac{\partial x}{\partial q_r} \frac{\partial y}{\partial p_t} \theta_t \frac{\partial F}{\partial p_r} + \frac{\partial x}{\partial p_r} \frac{\partial y}{\partial q_t} \theta'_t \frac{\partial F}{\partial p_r} + \frac{\partial x}{\partial p_0} \theta'_r \frac{\partial F}{\partial q_r} \frac{\partial y}{\partial p_r} + \frac{\partial x}{\partial p_0} \theta_r \frac{\partial F}{\partial q_r} \frac{\partial y}{\partial p_t} + \left( \frac{\partial x}{\partial q_0} \theta'_r \frac{\partial F}{\partial q_r} + \frac{\partial x}{\partial p_0} \theta_r \frac{\partial F}{\partial q_r} \right) \left( \frac{\partial y}{\partial q_t} \theta'_t \frac{\partial F}{\partial p_r} + \frac{\partial y}{\partial p_t} \theta_t \frac{\partial F}{\partial p_r} \right)$$

$$F(\xi) = 0$$

$$\bar{\xi}_r = \xi_r + \theta_r F$$

$$\frac{\partial F}{\partial \bar{\xi}_r} = \frac{\partial F}{\partial \xi_r} (1 - \frac{\partial F}{\partial \xi_0} \theta_r)$$

$$\frac{\partial x}{\partial \bar{\xi}_r} = \frac{\partial x}{\partial \xi_r} + \frac{\partial x}{\partial \xi_0} \theta_r \frac{\partial F}{\partial \xi_r} = \frac{\partial x}{\partial \xi_r} + \frac{\partial x}{\partial \xi_0} (\xi_0 - \xi_r) \frac{1}{F} \frac{\partial F}{\partial \xi_r}$$

$$\frac{\partial x}{\partial \xi_r} - \frac{\partial x}{\partial \xi_1} = (\xi_0 - \xi_1) \frac{\partial x}{\partial \xi_0} \frac{1}{F} \frac{\partial F}{\partial \xi_r}$$

$$\frac{\partial x}{\partial \xi_1} - \frac{\partial x}{\partial \xi_r} = (\xi_0 - \xi_r) \frac{\partial x}{\partial \xi_0} \frac{1}{F} \frac{\partial F}{\partial \xi_r}$$

$$\frac{\partial F}{\partial \xi_r} \text{ is ind. of } r.$$

$$\left( \frac{\xi_0 - \xi_r}{F} \right) \left( \frac{\partial x}{\partial \xi_0} \frac{\partial F}{\partial \xi_r} - \frac{\partial x}{\partial \xi_r} \frac{\partial F}{\partial \xi_0} \right) = 0$$

$$\delta F = \frac{\partial F}{\partial \xi_1} \delta \xi_1 = - \frac{\partial F}{\partial \xi_2} \delta \xi_2$$

$$\delta x = \frac{\partial x}{\partial \xi_1} \delta \xi_1 + \frac{\partial x}{\partial \xi_2} \delta \xi_2$$

$$\frac{\delta x}{\delta F} = \frac{\frac{\partial x}{\partial \xi_1}}{\frac{\partial F}{\partial \xi_1}} - \frac{\frac{\partial x}{\partial \xi_2}}{\frac{\partial F}{\partial \xi_2}}$$

$$[x, F] = \overline{[x, F]} + \frac{\partial F}{\partial p_0} \left\{ \frac{\partial x}{\partial q_r} \left( \frac{\partial F}{\partial p_r} \theta_0 + \frac{\partial F}{\partial q_0} \theta'_r \right) \right\} + \frac{\partial x}{\partial q_0} \frac{\partial F}{\partial p_r} \theta'_r + \frac{\partial x}{\partial p_r} \theta_r \frac{\partial F}{\partial q_0}$$

$$= \overline{[x, F]} + \frac{\partial F}{\partial p_0} \frac{\partial F}{\partial p_r} \frac{\partial x}{\partial q_r} \theta_0 + \frac{\partial F}{\partial p_r} \frac{\partial F}{\partial q_0} \frac{\partial x}{\partial q_r} \theta'_r - \frac{\partial F}{\partial q_0} \frac{\partial F}{\partial p_r} \frac{\partial x}{\partial p_r} \theta'_r$$

$$= \overline{[x, F]} + \left( \frac{\partial F}{\partial p_r} \frac{\partial x}{\partial q_r} - \frac{\partial F}{\partial q_r} \frac{\partial x}{\partial p_r} \right) \left( \frac{\partial F}{\partial p_0} \theta_0 + \frac{\partial F}{\partial q_0} \theta'_0 \right) - \frac{\partial F}{\partial q_r} \frac{\partial F}{\partial p_0} \frac{\partial x}{\partial p_r} \theta_0$$

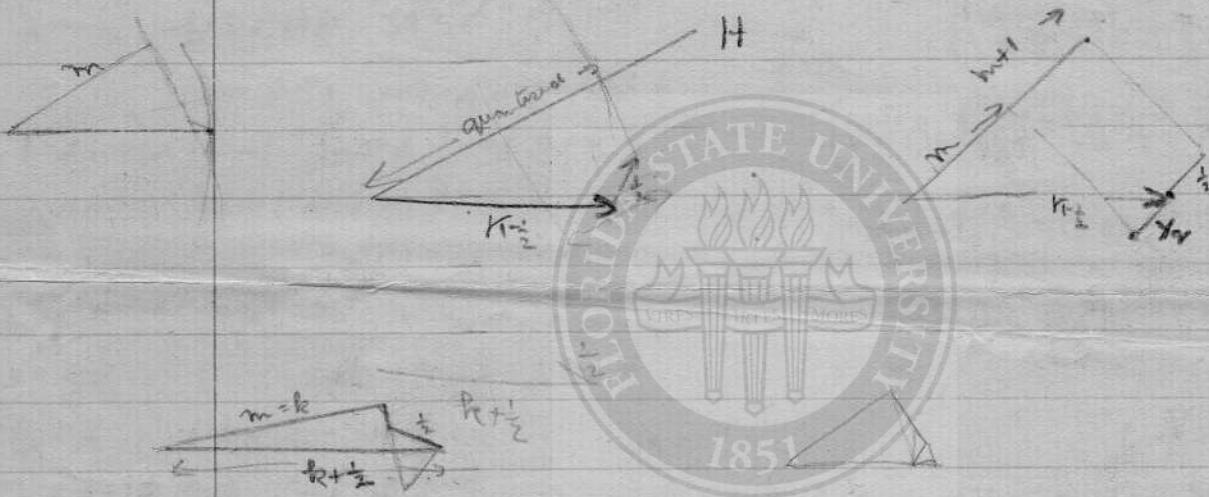
$$= \overline{[x, F]} \left\{ 1 + \frac{\partial F}{\partial p_0} \theta_0 + \frac{\partial F}{\partial q_0} \theta'_0 \right\}$$

$$\frac{\partial F}{\partial q_r} \frac{\partial F}{\partial p_r} \left( \frac{\partial x}{\partial q_0} \theta'_0 + \frac{\partial x}{\partial p_0} \theta_0 \right) \left( \frac{\partial y}{\partial q_t} \theta'_t + \frac{\partial y}{\partial p_t} \theta_t \right)$$

is constant, but will be increased by  $-eEx$ , and  $p_x$  will be increased by the same amount. Hence from (IV)

$$p_x =$$

The Hamiltonian equation thus still hold with the new Hamiltonian  $H'$ .  
Equation (III) can no longer be exactly true, since  $H$  can



$$(g'|\beta') = a(g'|\xi) e^{-\phi(g'|\xi)/\ell}$$

$$(\xi|\beta') = a e^{-\phi/\ell}$$

$a^2 dg'$  is the volume of  $\eta$ -space  
for which  $g' < g < g' + dg'$   
when  $\xi = \xi'$

This volume is symmetrical  
about the point  $\eta = \frac{\partial \phi}{\partial \xi'} = \eta_0$

$$\delta(g'|\xi') = (g'|\beta') (g'|\beta'')$$

$$\begin{aligned} & [i\hbar \delta(g'|\xi') + \delta(g'|\xi') i\hbar] (\xi'|\xi'') \\ &= -i\hbar \int [\delta(\xi' - \xi''') (\xi''|\beta') (g'|\beta'') + (g'|\beta') (g'|\beta'') \delta(\xi'' - \xi''')] d\xi''' \\ &= -i\hbar \left\{ \frac{\partial}{\partial \xi'} [(\xi'|\beta') (g'|\beta'')] - \frac{\partial}{\partial \xi''} [(g'|\beta') (g'|\beta'')] \right\} \\ &= -i\hbar (\xi'|\beta') (g'|\beta'') \left[ \frac{\partial}{\partial \xi'} \log(\xi'|\beta') - \frac{\partial}{\partial \xi''} \log(g'|\beta'') \right] \\ &= -i\hbar (\xi'|\beta') (g'|\beta'') \left[ \frac{\partial}{\partial \xi'} \log a(g'|\xi') - \frac{\partial}{\partial \xi''} \log a(g'|\xi'') \right] \\ &\quad + (\xi'|\beta') (g'|\beta'') \left[ \frac{\partial}{\partial \xi'} \phi(g'|\xi') + \frac{\partial}{\partial \xi''} \phi(g'|\xi'') \right] \end{aligned}$$

$$H_T = H_0 + \sum_n H(n) + \sum_{mn} V(mn)$$

$$V(r_1, r_2, \dots; a_1, a_2, \dots) = 0 \text{ except when 2 and only 2 } a_i \text{ differ from the corr. } r_i$$

$$= V_{r_m r_n} a_m a_n \text{ when } a_m \neq r_m, a_n \neq r_n \text{ and every other } a_i = r_i$$

$$\begin{aligned} \mathcal{L}(r_1, r_2, \dots) &= \dots + \sum_{mn} \sum_{a_m \neq r_m} \sum_{a_n \neq r_n} V_{r_m r_n} a_m a_n \mathcal{B}(r_1, r_2, \dots, a_m, a_n) \\ &= \dots + \sum_{a_m, a_n} \frac{N_{r_m} N_{r_n}}{2} [N_1! N_2! \dots (N_{r_m}-1)! (N_{r_n}-1)! \dots (N_{a_m+1})! (N_{a_n+1})! / N!]^{\frac{1}{2}} \end{aligned}$$

$$(ab)_1 = a_1 b_1 + a_2 b_2$$

$$(ab)_2 = a_1 b_2 + a_2 b_1$$

$$\bar{a}_1 = \lambda_1 a_1 + \lambda_2 a_2$$

$$\bar{a}_2 = \mu_1 a_1 + \mu_2 a_2$$

$$(\bar{ab})_1 = \lambda_1 (ab)_1 + \lambda_2 (ab)_2 = \lambda_1 (a_1 b_1 + a_2 b_2) + \lambda_2 (a_1 b_2 + a_2 b_1)$$

$$= (\bar{a}\bar{b})_1 = \bar{a}_1 \bar{b}_1 + \bar{a}_2 \bar{b}_2 = (\lambda_1 a_1 + \lambda_2 a_2)(\lambda_1 b_1 + \lambda_2 b_2) + (\mu_1 a_1 + \mu_2 a_2)(\mu_1 b_1 + \mu_2 b_2)$$

$$(\bar{ab})_2 = \mu_1 (ab)_1 + \mu_2 (ab)_2 = \mu_1 (a_1 b_1 + a_2 b_2) + \mu_2 (a_1 b_2 + a_2 b_1)$$

$$= (\bar{a}\bar{b})_2 = \bar{a}_1 \bar{b}_2 + \bar{a}_2 \bar{b}_1 = (\lambda_1 a_1 + \lambda_2 a_2)(\mu_1 b_1 + \mu_2 b_2) + (\mu_1 a_1 + \mu_2 a_2)(\lambda_1 b_1 + \lambda_2 b_2)$$

$$a_1 b_1 \quad \lambda_1 = \lambda_1^2 + \mu_1^2$$

$$a_1 b_2 \quad \lambda_2 = \lambda_1 \lambda_2 + \mu_1 \mu_2$$

$$a_2 b_1 \quad \lambda_2 = \lambda_1 \lambda_2 + \mu_1 \mu_2$$

$$a_2 b_2 \quad \lambda_1 = \lambda_2^2 + \mu_2^2$$

$$1) \mu_1 = 2\lambda_1 \mu_1$$

$$2) \mu_2 = \lambda_1 \mu_2 + \lambda_2 \mu_1$$

$$3) \mu_2 = \lambda_2 \mu_1 + \lambda_1 \mu_2$$

$$4) \mu_1 = 2\lambda_2 \mu_2$$

$$5) \lambda_1 = \frac{1}{2} \text{ or } \mu_1 = 0$$

$$\text{If } \mu_1 = 0, \text{ from 1) } \lambda_1 = 0 \text{ or } 1 \text{ from 2) } \lambda_2 = 0 \text{ or } \lambda_1 = 1$$

$$\text{from 6) } \mu_2 = 0 \text{ or } \lambda_1 = 1 \text{ from 8) } \lambda_2 = 0 \text{ or } \mu_2 = 0$$

$$\lambda_1 = 0, \lambda_2 = 0, \mu_2 = 0 \quad \left| \begin{array}{l} \lambda_1 = 1 \quad \mu_2 = 0 \quad \lambda_2 = \pm 1 \\ \text{or } \lambda_2 = 0 \quad \mu_2 = \pm 1 \end{array} \right.$$

$$\text{If } \lambda_1 = \frac{1}{2} \text{ from 1) } \mu_1 = \pm \frac{1}{2} \text{ from 2) } \lambda_2 = 2\mu_1 \mu_2 = \pm \mu_2$$

$$\text{from 6) } \mu_2 = 2\lambda_2 \mu_1 = \pm \lambda_2 \quad \mu_1 = \pm \mu_2^2$$

$$\text{from 4) } \frac{1}{2} = 2\lambda_2^2 \quad \text{from 7) } \pm \frac{1}{2} = \pm 2\lambda_2^2$$

$$\lambda_2^2 = \mu_2^2 = \frac{1}{4} \quad \mu_1 = \frac{1}{2}, \lambda_2 = \mu_1 = \frac{1}{2}$$

$$\mu_1 = -\frac{1}{2}, \lambda_2 = \mu_1 = -\frac{1}{2}$$

$$\bar{a}_1 = \frac{1}{2}(a_1 + a_2) \quad \bar{a}_2 = \frac{1}{2}(a_1 - a_2)$$

$$\bar{b}_1 = \frac{1}{2}(b_1 + b_2) \quad \bar{b}_2 = \frac{1}{2}(b_1 - b_2)$$

$$(\bar{ab})_1 = \frac{1}{2}(a_1 b_1 + a_2 b_2) + \frac{1}{2}(a_1 b_2 + a_2 b_1) = \frac{1}{2}(a_1 + a_2)(b_1 + b_2)$$

$$(\bar{ab})_2 = \frac{1}{2}(a_1 b_1 + a_2 b_2) - \frac{1}{2}(a_1 b_2 + a_2 b_1)$$

$$\mathcal{B}(N_1, N_2, \dots, N_{r_m-1}, N_{r_m}, N_{r_m+1}, \dots, N_{a_m+1}, N_{a_n+1}, \dots) V_{r_m r_n} a_m a_n$$

$$a_{n,n+1} \psi_{n+1} = \lambda \psi_n$$

$$\frac{\psi_{n+1}}{\psi_n} = \frac{\lambda}{a_{n,n+1}}$$

Use only incomplete for system satisfies

$$F \psi_n = 0$$

$$\sum_m W_{nm} \psi_m = \lambda \psi_n$$

$$a_{nm} = \delta_{nm} + \epsilon_{nm}$$

$$\tilde{a}_{11} \tilde{a}_{11} + \tilde{a}_{12} \tilde{a}_{21}$$

$$= (\lambda - W_n) \psi_n = \sum_m \epsilon_{nm} \psi_m$$

$$\tilde{a}_{11} \tilde{a}_{11} g_{11} + \tilde{a}_{11} \tilde{a}_{12} g_{21} + \tilde{a}_{12} \tilde{a}_{21} g_{11} + \tilde{a}_{12} \tilde{a}_{22} g_{21}$$

$$\lambda = W_n + \delta$$

$$\sum_m \epsilon_{nm} \psi_m = 0$$

$$\tilde{a}_{11} g_{11} \tilde{a}_{21} g_{22}$$

all  $\psi$ 's are all except  $\psi_k$

$$\sum_k \epsilon_{nk} = 0$$

$$= \alpha \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

(11)

(21)

$$\sum_r b_r(\alpha) c_r(\beta)$$

$$\begin{vmatrix} a_{11}x_{11} + a_{12}x_{12} + a_{21}x_{21} + a_{22}x_{22} \\ c_{11}x_{11} + c_{12}x_{12} + c_{21}x_{21} + c_{22}x_{22} \end{vmatrix}$$

$$\begin{vmatrix} b_{11}x_{11} + b_{12}x_{12} + b_{21}x_{21} + b_{22}x_{22} \\ d_{11}x_{11} + d_{12}x_{12} + d_{21}x_{21} + d_{22}x_{22} \end{vmatrix}$$

$$x_{11} x_{12}$$

$$x_{11} x_{12}$$

$$\checkmark a_{11} d_{11} = b_{11} c_{11}$$

$$a_{11} d_{12} + a_{12} d_{11} - b_{11} c_{12} - b_{12} c_{11} = 0$$

$$a_{12} d_{12} = b_{12} c_{12}$$

$$a_{22} d_{12} + a_{21} d_{11} - b_{21} c_{12} - b_{22} c_{11} = 0$$

$$a_{11} d_{21} = b_{11} c_{21}$$

$$a_{11} d_{22} + a_{12} d_{21} - b_{11} c_{22} - b_{12} c_{21} = 0$$

$$a_{12} d_{21} = b_{12} c_{21}$$

$$a_{12} d_{22} + a_{21} d_{21} - b_{12} c_{22} - b_{21} c_{21} = 0$$

$$a_{21} d_{21} = b_{21} c_{21}$$

$$a_{21} d_{22} + a_{22} d_{21} - b_{21} c_{22} - b_{22} c_{21} = 0$$

$$a_{22} d_{22} = b_{22} c_{22}$$

$$a_{22} d_{22} + a_{21} d_{21} - b_{22} c_{22} - b_{21} c_{21} = 0$$

$$\frac{a_{12}}{b_{12}} = \frac{c_{12}}{d_{12}} = x_{12}$$

$$a_{11} = b_{11} g_{11}$$

$$b_{11} = \frac{2}{1} - 1$$

$$b_{11} = b_{21} g_{11}$$

$$r_{11} = r_{11}$$

$$c_{11} = b_{11} g_{12}$$

$$a_{12} = b_{11} g_{21}$$

$$d_{11} = b_{21} g_{12}$$

$$a_{22} = b_{12} g_{21}$$

$$b_{21} = b_{21} g_{21}$$

$$c_{22} = b_{12} g_{22}$$

$$d_{22} = b_{12} g_{22}$$

$$a_{12} b_{21} g_{12} - b_{12} b_{11} g_{12} - c_{12} b_{21} g_{11} + d_{12} b_{11} g_{11} = 0$$

$$(b_{21} a_{12} - b_{11} b_{12}) g_{12} + (b_{11} d_{12} - b_{21} c_{12}) g_{11} = 0$$

$$(b_{21} x_{12} - b_{11}) (b_{12} g_{12} - d_{12} g_{11}) = 0$$

$$(b_{22} x_{12} - b_{12}) (b_{12} g_{22} - d_{12} g_{21}) = 0$$

$$t' = t - \frac{v}{c} x$$

$$J_1, J_2, \dots, W$$

$$w_1, w_2, \dots, t'$$

$$t_1' = t' + \alpha \sin v t'$$

$\alpha, v$  c-numbers

$$W_1 = W \frac{1}{1 - \alpha \sin v t'}; [W_1, t_1] = [W, t_1] \frac{1}{1 - \alpha \sin v t'} = (1 - \alpha \sin v t')^{-1} W$$

$$J_{r1} = J_r \frac{1}{1 - \alpha \sin v t'}$$

$$w_{r1} = w_r (1 - \alpha \sin v t')$$

$$A = \frac{d^{k+1}}{dx^{k+1}} (1-x)^k; B = \frac{d^{k+1}}{dx^{k+1}} (1-x)^{k+1}; C = \frac{d^{k+1}}{dx^{k+1}} (1-x)^{k+2}$$

$$B = \frac{d^{k+1}}{dx^{k+1}} \frac{d^2}{dx^2} [(1-x)^{k+1} (1-x)^2]$$

$$J = q$$

$$w = -p$$

$$\bar{J}_r = J_r$$

$$H_0(q) \frac{1}{1 - \alpha \sin v t'} \neq \frac{\partial S}{\partial t_1} = 0$$

$$S = H_0 f(t_1) + \int dr q_r$$

$$q_r = q_r; p_r = q_r$$

$$p_r = \frac{\partial H_0}{\partial q_r} f(t_1) + dr$$

$$p'(t_1) = \frac{1}{1 - \alpha \sin v t'}$$

$$\bar{w}_r = -\alpha_r = -p_r + \frac{\partial H_0}{\partial q_r} f(t_1) = w_r + \frac{\partial H_0}{\partial q_r} f(t_1)$$

$$[\bar{w}_1, \bar{w}] = -\frac{\partial H_0}{\partial J_r} p'(t_1) + \frac{\partial H_0}{\partial J_r} [p(t_1), w]$$

$$\bar{W} = W - H_0 f'(t_1)$$

$$\bar{W} = W + f(J_r)$$

$$\bar{J}_r = J_r$$

$$\bar{w}_r = w_r + \frac{\partial f}{\partial J_r}$$

In physical interpretation, instead of substituting c-number values, substitute c-equation i.e., equation disagreeing with quantum conditions.

$$We^{i\alpha t} = e^{i\alpha t} W$$

If  $t$  comes only through cive would Hamiltonian eqn be consistent with Q conditions?

$$ab + ba = 0 \quad a \text{ diagonal}$$

$$a_{mn} b_{mn} + b_{mn} a_{nn} = 0$$

$$b_{mn} = 0 \quad \text{or} \quad a_{mn} = -a_{nn}$$

$$K\alpha + \alpha K = 0$$

$$\alpha_{11} = -\alpha_{22}$$

$$0 = K_{11} = K_{22}$$

$$H(w) - wH = \bar{w}$$

$$H(w + e) - (w + e)H = \bar{w}$$

$$F = (i\hbar \frac{\partial}{\partial x} + K_x)^2 + \dots$$

$$\phi(f, g) = \sum_{mn} a_{mn} f_m \bar{g}_n$$

$$\phi(f, Tg) = \sum_{mnk} f_m a_{mn} T_{nk} \bar{g}_k$$

$$\phi(T^* f, g) = \sum_{mnk} T_{mk}^* f_n a_{nk} \bar{g}_k$$

$$a_{nm} T_{mk} = T_{mn}^* a_{nk}$$



$$g g = \int (x \cdot p)$$

$$H = \int (x \cdot p \cdot x \cdot p)$$

$$(g g + \varepsilon H - W) \psi_n = 0$$

$$(g g - W) \psi_n' = 0$$

$$(W_1 + H) \psi_n' = (g g - W) \psi_n''$$

$$(T_2 + \varepsilon H_1 + \varepsilon^2 V_2) \psi = 0$$

$$T_2 \psi_n'' + (H_1 - W) \psi_n' = 0$$

$$\psi = \psi_n' + \varepsilon \psi_n''$$

$\psi_n'$  is fn of  $x_i$ 's only

$$\psi = \psi_1, \psi_2$$

$\psi_n$  mainly ind. of  $x_1, x_2, x_3$

$\psi_1, \psi_2$  small

$$B \psi_k = \sum_l b_{elk} \psi_l$$

$$\psi_k = \sum_l a_{kel} \bar{\psi}_l$$

$$\sum_l a_{kel} a_{elm} = \delta_{km}$$

$$B \sum_l a_{kel} \bar{\psi}_l = \sum_m b_{elm} a_{elm} \bar{\psi}_m$$

$$B \bar{\psi}_l = \sum_m \bar{b}_{lm} \bar{\psi}_m$$

$$B \sum_l a_{kel} \bar{\psi}_l = \sum_m \bar{b}_{lm} a_{elm} \bar{\psi}_m$$

$$\psi_n = \psi_n' + \varepsilon \psi_n''$$

$$\psi = \psi_n' + \varepsilon \psi_n''$$

$$W_n \psi_n' - W \psi_n' + \varepsilon H \psi_n' + \varepsilon (g g - W) \psi_n'' = 0$$

$$H \psi_n' + (g g - W) \psi_n'' = 0$$

Boundary values are  $f(y) = 0$

$$\delta(g(y) - g') = \delta(y - y') / \frac{\partial g(y)}{\partial y'} = \int e^{i(y-y')n} dn / \frac{\partial g}{\partial y'}$$

Suppose  $g(y) = g'$

$$\int \{ a(x) b(x'') dx'' \delta(x'' - x''') - \delta'(x' - x'') a(x'') b(x''') \}$$

$$= -a(x') \frac{\partial b(x'')}{\partial x'''} - \frac{\partial a(x'')}{\partial (x')} b(x''') = -\left( \frac{\partial}{\partial x'} + \frac{\partial}{\partial x'''} \right) [a(x'') b(x''')] = 0$$

$$\theta(x'x'') = \delta'(x' - x'')$$

$$\phi(x'x'') = a(x') b(x'')$$

$$(\theta\phi + \phi\theta)(x'x'') = \int \{ \delta'(x' - x''') dx''' a(x''') b(x'') + a(x') b(x''') dx''' \delta(x'' - x''') \}$$

$$= \frac{\partial a(x')}{\partial x'} b(x'') - a(x') \frac{\partial b(x'')}{\partial x''}$$

$$(\theta\phi + \phi\theta)(x'x') = \frac{\partial a(x')}{\partial x'} b(x') - a(x') \frac{\partial b(x')}{\partial x'} = a(x') b(x') \frac{\partial}{\partial x'} \log \frac{a(x')}{b(x')}$$

The  $x$ 's and their time con. conjugates form a system in which an arb. q. number is not a single valued function.

$$\delta'(x' - x'') \frac{\sqrt{a(x') b(x'')}}{a(x'') b(x'')}} = \theta(x'x'')$$

$$\neq \delta'(x' - x'') \frac{\sqrt{a(x')}}{a(x'')} \frac{\sqrt{b(x'')}}{b(x'')} = \theta(x'x'')$$

$$a(x') b(x'') = \phi(x'x'')$$

$$(\theta\phi + \phi\theta)(x'x'') = + \frac{\partial}{\partial x'''} \frac{\sqrt{a(x'')}}{a(x'')} \frac{\sqrt{b(x''')}}{b(x''')} a(x'') b(x''') \quad x''' = x''$$

$$= a(x') \frac{\partial}{\partial x'''} b(x''') \frac{\sqrt{a(x'')}}{a(x'')} \frac{\sqrt{b(x''')}}{b(x''')} \quad x''' = x''$$

$$= b(x'') \sqrt{\frac{a(x')}{b(x'')}} \frac{\partial}{\partial x'} \sqrt{\frac{a(x'')}{b(x'')}} = a(x') \sqrt{\frac{b(x'')}{a(x'')}} \frac{\partial}{\partial x'} \sqrt{\frac{a(x'')}{b(x'')}} = 0$$

$$(\theta\phi + \phi\theta)(x'x') = 0$$

matrix  
not an  $x''$   
[  $\frac{\partial}{\partial x'} K_2$  must be a matrix  
with central symmetry like  $K_2$  ]



$$\Sigma (p_r - \frac{e}{c} K_r)^2 = 0$$

+ ~~the same~~ ~~the same~~

$$\Sigma (-i\hbar \frac{\partial}{\partial q_r} - \frac{e}{c} K_r)^2 \Psi_K = 0$$

$$\bar{\Psi}_{K_n} = \Psi_{-K_n} \text{ one of}$$

$$\Sigma (-i\hbar \frac{\partial}{\partial q_r} + \frac{e}{c} K_r) \bar{\Psi}_K = 0$$

Do the  $\Psi_K$  corresponding to +ve energy and the  $\Psi_{-K}$  satisfy orthogonal relations

$$\int_{-\infty}^{\infty} \Psi'_{K_W} \bar{\Psi}'_{K'_W} dW = 0$$

(2.20)

|                                  |                 |
|----------------------------------|-----------------|
| $e^{-iWt/\hbar}$                 | $e^{iWt/\hbar}$ |
| $\Psi_{K_n} = \bar{\Psi}_{-K_n}$ | $\Psi_{K,-n}$   |
| $\bar{\Psi}_{K,-n}$              | $\Psi_{K_n}$    |
| $= \Psi_{-K,-n}$                 | $= \Psi_{-K,n}$ |

$$= \int_0^{\infty} \Psi'_{K_W} \bar{\Psi}'_{K'_W} dW + \int_{-\infty}^0 \Psi'_{-K,-W} \bar{\Psi}'_{-K,-W} dW$$

We want

$$\int_0^{\infty} (\Psi'_{K_W} \bar{\Psi}'_{K'_W} + \Psi'_{-K,-W} \bar{\Psi}'_{-K,-W}) dW$$

$$\text{or } \int_0^{\infty} \{x'(W) \frac{d}{dW} (W/x(W)) + (x''(W)) (W/x'(W))\} dW$$

$\Psi_{-K,-W}$  refers to same equations of motion as  $\Psi_{K,W} = \bar{\Psi}_{-K,-W}$

$$\int \{ (q'(p')) (p'(q')) + (q''(p')) (p''(q')) \} dp' = 0 (?)$$

for the energy

$$\int (q'(p')) (p'(q')) dp' = \text{pure imaginary} (?)$$

$$= \hbar [i(q' - q'')]$$

$$(1/1)(1/1) + (1/2)(2/2) = 0 \quad \alpha \neq \alpha'$$

$$\frac{(1/1)}{(1/2)} = - \frac{(2/2)}{(1/1)}$$

$$\int (\alpha'(p')) dq' (q'(p')) = \delta(\alpha' - \alpha'')$$

$$\int (\alpha'(p')) dq' \int (\alpha''(p'')) dq'' (\alpha''(p'')) = \int \delta(\alpha' - \alpha'') dq'' (\alpha''(p'')) = (\alpha'(p')) \text{ when } \alpha' > 0$$

$$= 0 \quad \alpha' < 0$$

$$\int (\alpha'(p')) \left( \int (\alpha''(p'')) dq'' - \delta(\alpha' - \alpha'') \right) dq' = 0 \quad \alpha' > 0$$

$$= -(\alpha'(p')) \quad \alpha' < 0$$

$$\int (\alpha'(p')) (\alpha'(q'')) dq' dq'' \int (\alpha''(p'')) dq'' (\alpha''(p'')) = \int (\alpha'(q'')) dq'' \quad \alpha' > 0$$

$$= 0 \quad \alpha' < 0$$

should vanish (to 1st order in  $\hbar$ ) if  $\int$  is to be antisym. in  $q' \alpha' q''$ .

$$(\alpha'(p'))^2 = (\alpha'(p')) \text{ with } \hbar \text{ neglected.}$$

$$y_1 - y_2 = a + ib$$

$$y_1 = a + ib$$

$$y_2 = a + ib'$$

$$(p+c)K=0$$

$$(i\hbar \frac{\partial}{\partial t} + c)K=0$$

$$\Delta \dot{K}_q = -\hbar \left( \hbar \frac{\partial K_0}{\partial x_1 \partial x_2} \right) + \left( \frac{\partial^2 K_0}{\partial x_1 \partial x_2} \right) \hbar$$

$$\Delta \dot{x}_q = \hbar \frac{\partial K_0}{\partial x_1}$$

$$\sum_{kl} \left[ \int_0^T K' e^{i(\omega_k - \omega_l)t} dt \int_0^t K' e^{i(\omega_l - \omega_k)s} ds \right] \dot{y}_{mn} a_k a_m \dot{y}_{nk}$$

$$\sum_n \left[ \left( \hbar \omega - \frac{e^2}{c} K_V \right)^2 + m^2 c^2 \right] = 0$$

$$\left( \hbar^2 + \frac{e^2}{c^2} K_V^2 + m^2 c^2 \right)^2 = 4 \frac{e^2}{c^2} (\hbar K_V)^2$$

$$\hbar^2 K_0^2 - 2 \frac{e^2}{c^2} \hbar^2 K_0^2$$

$$\hbar^4 K_0^2 + 2 \frac{e^2}{c^2} \hbar^2 K_0^2 + \frac{e^4}{c^4} K_V^2 K_0^2 + 2 m^2 c^2 \left( \hbar^2 + \frac{e^2}{c^2} K_V^2 \right) + m^4 c^4 - 4 \frac{e^2}{c^2} \hbar K_V \hbar K_0 K_0 = 0$$

$$[a, b] = \frac{2e}{c} \left[ \hbar^2 + \frac{e^2}{c^2} K_V^2, \hbar K_0 \right] = \frac{2e}{c} \left\{ \hbar \left( \hbar \frac{\partial K_0}{\partial x_1} + \frac{\partial K_0}{\partial x_1} \hbar \right) + \frac{e^2}{c^2} \frac{\partial K_V^2}{\partial x_0} K_0 \right\}$$

$$a^2 - b^2 = (a+b)(a-b) + i\hbar [b, a] = (a+b) \left\{ a-b + \frac{i\hbar [b, a]}{a+b} \right\}$$



$$\dot{y}_{mn} a_k a_m \dot{y}_{nk}$$

$$a = \hbar^2 + \frac{e^2}{c^2} K_V^2 + m^2 c^2 \quad b = \frac{2e}{c} \hbar K_V$$

$$[a, b]$$

$$\int_0^{\infty} \cos(\xi' \eta') \cos(\xi'' \eta') d\eta' = \xi' [\delta(\xi' - \xi'') + \delta(\xi' + \xi'')]$$

$$(\xi''/\eta') = \begin{cases} \cos(\xi' \eta') & \xi' > 0 \quad \eta' > 0 \\ \sin(\xi' \eta') & \xi' < 0 \quad \eta' < 0 \end{cases}$$

$$(\eta'/\xi') = \begin{cases} \cos(\xi' \eta') & \eta' > 0 \\ \sin(\xi' \eta') & \eta' < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} (\xi' \eta') (\eta' \xi'') d\eta' = \int_{-\infty}^{\infty} \cos(\xi' \eta') \begin{pmatrix} \cos(\xi'' \eta') & \eta' > 0 \\ \sin(\xi'' \eta') & \eta' < 0 \end{pmatrix} d\eta'$$



$$\frac{1}{2} J_2^{\frac{1}{2}} e^{i\theta} J_2^{\frac{1}{2}} (\lambda_2 = i [J, \lambda_2]) = \frac{1}{4} (J_2^2 - M_2^2)^{\frac{1}{2}} \left\{ - (J_2 - k + h)^{\frac{1}{2}} (J_2 + k + h)^{\frac{1}{2}} - (J_2 - k_2 + h)^{\frac{1}{2}} (J_2 + k_2 + h)^{\frac{1}{2}} - \frac{1}{2} k^{\frac{1}{2}} e^{-i\theta} k^{-\frac{1}{2}} \right. \\ \left. + (J_2 + k + h)^{\frac{1}{2}} (J_2 + k - h)^{\frac{1}{2}} (J_2 + k_2 + h)^{\frac{1}{2}} (J_2 + k_2 - h)^{\frac{1}{2}} k^{\frac{1}{2}} e^{-i\theta} k^{-\frac{1}{2}} \right\} \quad 20.$$

$$(J + k)^{\frac{1}{2}} J^{\frac{1}{2}} (\lambda_2 = i [J, \lambda_2]) = \frac{1}{4} (J^2 - M_2^2)^{\frac{1}{2}} - F_{+1} \frac{J_1}{k^{\frac{1}{2}} (k-h)^{\frac{1}{2}}} e^{i(\theta-\psi)} + F_{+1} \frac{J_1}{k^{\frac{1}{2}} (k+h)^{\frac{1}{2}}} e^{i(\theta+\psi)}$$

which gives

$$F_{+1} = \frac{1}{4} (J + k + h + \frac{1}{2} h)^{\frac{1}{2}} (J + k - h + \frac{1}{2} h)^{\frac{1}{2}} (J + k + h + \frac{1}{2} h)^{\frac{1}{2}} (J + k - h + \frac{1}{2} h)^{\frac{1}{2}} / (J + \frac{1}{2} h)^{\frac{1}{2}} k^{\frac{1}{2}} (k+h)^{\frac{1}{2}}$$

$$F_{+1}' = \frac{1}{4} (J - k + h' + \frac{1}{2} h)^{\frac{1}{2}} (J + k + h' - \frac{1}{2} h)^{\frac{1}{2}} (J - k + h' + \frac{1}{2} h)^{\frac{1}{2}} (-J + k + h' - \frac{1}{2} h) / (J + \frac{1}{2} h)^{\frac{1}{2}} k^{\frac{1}{2}} (k-h)^{\frac{1}{2}}$$

$$\left\{ \lambda_2 - i [J, \lambda_2] = \frac{1}{4} (J_2^2 - M_2^2)^{\frac{1}{2}} \frac{J_1^{\frac{1}{2}}}{J^{\frac{1}{2}} (J + \frac{1}{2} h)^{\frac{1}{2}} k^{\frac{1}{2}} (k+h)^{\frac{1}{2}}} F_{+1} e^{-i(\theta+\psi)} - \frac{J_1^{\frac{1}{2}}}{J^{\frac{1}{2}} (J + \frac{1}{2} h)^{\frac{1}{2}} k^{\frac{1}{2}} (k-h)^{\frac{1}{2}}} F_{+1}' e^{i(\theta-\psi)} \right\} \quad (53)$$

similarly it may be shown that

$$\lambda_2 + i [J, \lambda_2] = \frac{1}{4} (J_2^2 - M_2^2)^{\frac{1}{2}} \frac{J_2^{\frac{1}{2}}}{J^{\frac{1}{2}} (J - \frac{1}{2} h)^{\frac{1}{2}} k^{\frac{1}{2}} (k+h)^{\frac{1}{2}}} F_{-1} e^{i(\theta+\psi)} - \frac{J_2^{\frac{1}{2}}}{J^{\frac{1}{2}} (J - \frac{1}{2} h)^{\frac{1}{2}} k^{\frac{1}{2}} (k-h)^{\frac{1}{2}}} F_{-1}' e^{-i(\theta-\psi)} \quad (56)$$

where  $F_{-1}'$  is the quantity obtained by writing  $-h$  for  $h$  in  $F_{+1}$  and  $F_{-1}$  the quantity obtained in the same way from  $F_{+1}'$

$$F_{-1}' = \frac{1}{4} (J + k + h' - \frac{1}{2} h)^{\frac{1}{2}} (J + k - h' - \frac{1}{2} h)^{\frac{1}{2}} (J + k + h' - \frac{1}{2} h)^{\frac{1}{2}} (J + k - h' - \frac{1}{2} h)^{\frac{1}{2}} / (J - \frac{1}{2} h)^{\frac{1}{2}} k^{\frac{1}{2}} (k-h)^{\frac{1}{2}}$$

$$F_{-1} = \frac{1}{4} (J - k + h' - \frac{1}{2} h)^{\frac{1}{2}} (-J + k + h' + \frac{1}{2} h)^{\frac{1}{2}} (J - k + h' - \frac{1}{2} h)^{\frac{1}{2}} (-J + k + h' + \frac{1}{2} h)^{\frac{1}{2}} / (J - \frac{1}{2} h)^{\frac{1}{2}} k^{\frac{1}{2}} (k+h)^{\frac{1}{2}}$$

and also

$$q \cdot h = \frac{1}{2} \left\{ (J + k + h' - \frac{1}{2} h)^{\frac{1}{2}} (J + k - h' - \frac{1}{2} h)^{\frac{1}{2}} (J - k + h' + \frac{1}{2} h)^{\frac{1}{2}} (J + k + h' - \frac{1}{2} h)^{\frac{1}{2}} k^{-\frac{1}{2}} e^{i\theta} k^{\frac{1}{2}} \right. \\ \left. + (J + k + h' + \frac{1}{2} h)^{\frac{1}{2}} (J + k - h' + \frac{1}{2} h)^{\frac{1}{2}} (J - k + h' - \frac{1}{2} h)^{\frac{1}{2}} (-J + k + h' + \frac{1}{2} h)^{\frac{1}{2}} k^{-\frac{1}{2}} e^{-i\theta} k^{\frac{1}{2}} \right\} \quad \text{omit}$$

$$= \left( \frac{J_1 J_2}{J} \right)^{\frac{1}{2}} F_0 e^{-i\theta} + \left( \frac{J_1 J_2}{J} \right)^{\frac{1}{2}} F_0' e^{i\theta} \quad (57)$$

where

$$F_0 = \frac{1}{2} (J + k + h' + \frac{1}{2} h)^{\frac{1}{2}} (J + k - h' + \frac{1}{2} h)^{\frac{1}{2}} (J - k + h' - \frac{1}{2} h)^{\frac{1}{2}} (-J + k + h' + \frac{1}{2} h)^{\frac{1}{2}} J^{\frac{1}{2}} / (J J_2)^{\frac{1}{2}} k^{\frac{1}{2}} (k+h)^{\frac{1}{2}}$$

and  $F_0'$  is the quantity obtained by writing  $-h$  for  $h$  in  $F_0$

$$F_0' = \frac{1}{2} (J + k + h' - \frac{1}{2} h)^{\frac{1}{2}} (J + k - h' - \frac{1}{2} h)^{\frac{1}{2}} (J - k + h' + \frac{1}{2} h)^{\frac{1}{2}} (-J + k + h' - \frac{1}{2} h)^{\frac{1}{2}} J^{\frac{1}{2}} / (J J_2)^{\frac{1}{2}} k^{\frac{1}{2}} (k-h)^{\frac{1}{2}}$$

$$z = \frac{M_2}{J_1 J_2} q \cdot h' + \frac{1}{2} i \frac{1}{J_1} (\lambda_2 - i [J, \lambda_2]) - \frac{1}{2} i \frac{1}{J_2} (\lambda_2 + i [J, \lambda_2])$$

$$= \frac{M_2}{J_1 J_2} (F_{+1} e^{-i\theta} + F_{+1}' e^{i\theta}) + \frac{1}{2} i \frac{(J + M_2 + \frac{1}{2} h)(J + M_2 + \frac{1}{2} h)^{\frac{1}{2}}}{J^{\frac{1}{2}} (J + \frac{1}{2} h)^{\frac{1}{2}} k^{\frac{1}{2}} (k+h)^{\frac{1}{2}}}$$

$$j(j-1)(\mu_2 + i[j, \mu_2])(\lambda_2 - i[j, \lambda_2]) = (j_2^2 - \mu_2^2)(\mu_2 \lambda_2 + \mu_2 \lambda_2 + \mu_2 \lambda_2) + i h M_2 (\mu_2 \lambda_2 - \mu_2 \lambda_2) - i j_2 \{ M_2 (\mu_2 \lambda_2 - \mu_2 \lambda_2) + M_2 (\mu_2 \lambda_2 - \mu_2 \lambda_2) \} \quad (55)$$

now  $\mu_2 \lambda_2 + \mu_2 \lambda_2 + \mu_2 \lambda_2 = \sum_{m_2} (m_2 m_2' - m_2 m_2') (M_2 z - M_2 z - i h \mu_2)$

Put dashed letters first  $\rightarrow \sum_{m_2} (m_2 m_2' - m_2 m_2') (M_2 z - M_2 z - i h \mu_2) = \sum_{m_2} \{ m_2' (M_2 M_2 + M_2 M_2 + M_2 M_2) - (M_2' M_2 + M_2' M_2 + M_2' M_2) m_2 \}$

$= m_2 M_2 m_2' q + \sum_{m_2} \{ m_2' (m_2' M_2 - M_2 m_2') + m_2 (m_2' M_2 - M_2 m_2') \} z$

$= m_2 M_2 m_2' q + i h \mu_2 q$

using (5). Also

$$[\lambda_2, \mu_2] = [M_2 z - y M_2, \mu_2] = M_2 [z, \mu_2] - [y, \mu_2] M_2 + y \mu_2$$

$$= M_2 (-y m_2') - (x m_2' + z m_2') M_2 + y (M_2 m_2' - m_2' M_2) = -q m_2' M_2 - M_2 q m_2'$$

$$\mu_2 \lambda_2 - \mu_2 \lambda_2 = \mu_2 (x M_2 - M_2 x) - \mu_2 (M_2 z - y M_2)$$

$$= (\mu_2 x + \mu_2 y + \mu_2 z) M_2 - (\mu_2 M_2 + \mu_2 M_2 + \mu_2 M_2) z = \mu_2 q M_2$$

and similarly  
while  
and

$$\mu_2 \lambda_2 - \mu_2 \lambda_2 = \mu_2 q M_2 \quad \mu_2 \lambda_2 - \mu_2 \lambda_2 = \mu_2 q M_2$$

$$\mu_2 q = \sum_{m_2} (m_2' x - m_2 y) m_2' = k [k, q, m_2] - \frac{1}{2} i h q m_2'$$

from (31). Using these results, and the fact that  $\mu_2 q$  commutes with  $M_2$ ,  $\mu_2$  and  $M_2$ , eqn (55) becomes  $(M_2 \mu_2 M_2)$  commutes with  $\mu_2 q$  (since they commute with  $k$  and  $q m_2'$ )

$$j(j-1)(\mu_2 + i[j, \mu_2])(\lambda_2 - i[j, \lambda_2]) = (j_2^2 - \mu_2^2) (M_2 m_2' + i h \mu_2 q) + i h M_2^2 - j_2^2 (M_2^2 + M_2^2) \mu_2 q$$

$$= (j_2^2 - M_2^2) (M_2 m_2' + i h \mu_2 q - i j_2 \mu_2 q)$$

$$= \frac{1}{2} (j_2^2 - M_2^2) \{ (j_2 j_2 + k^2 - k'^2) \mu_2 - 2 i j_2 (k [k, q, m_2] - \frac{1}{2} i h q m_2') \}$$

$$= \frac{1}{4} (j_2^2 - M_2^2) \{ (k + k' + j_2)(k + k' - j_2) (q m_2' + i [k, q, m_2]) + (k + k' + j_2)(k - k' + j_2) (q m_2' - i [k, q, m_2]) \}$$

Hence, substituting for  $(\mu_2 + i[j, \mu_2])$ ,  $(q m_2' + i [k, q, m_2])$  and  $(q m_2' - i [k, q, m_2])$  from - we find

$$(j + k) j (\lambda_2 - i[j, \lambda_2]) = \frac{1}{4} (j_2^2 - M_2^2) (j_1 - k + k')^2$$

after slight rearrangement of the order of factors

$$H = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) - \left( \frac{W}{c} + \frac{e^2}{cr} \right)^2 + m^2 c^2 = 0$$

$$\frac{1}{2m} \left( p^2 + \frac{p_1 p_2}{r^2} \right) - \frac{1}{2m} \left( \frac{W}{c} + \frac{e^2}{cr} \right)^2 + \frac{m^2 c^2}{2} = 0$$

$$\frac{1}{2m} \left( p^2 + \frac{p_1 p_2}{r^2} \right) - \frac{e^2}{r} - W = 0$$

$$\bar{p}^2 = p^2 - \frac{e^4}{c^2}$$

$$\bar{e}^2 = \frac{W}{mc^2} e^2$$

$$\bar{W} = -\frac{1}{2} mc^2 + \frac{1}{2} \frac{W^2}{mc^2}$$

We know the first transform from

$$\bar{x} \bar{y} \bar{z} \bar{W} \text{ to } J'_0 \bar{W}$$

and hence we know the second transform from

$$x y z W \text{ to } J'_0 W$$

$$(x'w | J'w'') = X(x'J' | w'') \delta(w' - w'')$$

$$\iint (J''w''' | x'w') (x'w' | J'w'') dn' dW' = \int \bar{X}(x'J' | w'') \bar{X}(x'J' | w''') \delta(w' - w''') dn'$$

$$\iint (x'w' | J'w'') (J'w'' | x''w''') dJ' dW'' = \delta(J' - J'') \delta(w' - w''')$$

$$= \int X(x'J' | w') dJ' \bar{X}(x''J' | w'') \delta(w' - w'') = \delta(x' - x'') \delta(w' - w'')$$

$$(x' | J') = (x' | J' | w') = X(x'J' | w') e^{i w' t' / \hbar}$$

$$\int (J''x') dx' (x' | J') = \int \bar{X}(x'J' | w'') dn' X(x'J' | w'') \delta(w' - w'')$$

$W$  is true canonical conjugate of  $t$ .

$$(a'g' | e'g'') = (a' | e') \delta(g' - g''). \text{ For such a transf. the canonical conjugates of the } g \text{ are not changed.}$$

$$(a'g' | e'g'') = \int \int (a'g' | e'g'') da' dg' f(g') \delta(e'g' - g'') / (e'g' | a'g'')$$

$$= \iint (a'g' | e'g'') da' dg' f(g') (e'g' | a'g'') = f(g') \delta(g' - g'') \delta(a' - a'')$$

$$x_0 \psi(x) = \sum_n x_{nn} \psi_n(x)$$

$$x_{nn} = \langle \psi | x | \psi \rangle$$

$$\sum_n P_{nn} P_{nn} = x_0 P_{nn}$$

$$x_0 P_{nn} = \sum_n P_{nn} x_{nn}$$

$$P_{nn} = \psi_n(x_0)$$

$$P_{nn} x_0 = \sum_n P_{nn} x_{nn} = \{P\} \{x\}$$

$$\{P\} x_0 = \{P\} \{x\}$$

$$P_{nn} = \psi_n(x_0)$$



$$0 < x < \pi$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x}$$

$$\text{begin for } \frac{1}{\pi} \sin nx$$

$$x_{nm} = \frac{2}{\pi} \int_0^{\pi} x \sin nx \sin mx dx = \frac{1}{\pi} \int_0^{\pi} x [m(m-n)x - m(m+n)x] dx$$

$$(m \neq n) = \frac{1}{\pi} \int_0^{\pi} x \left[ \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right] dx = 0 \quad m+n, \text{ even}$$

$$p(x) = \delta(x-a) \text{ except when } x \text{ and } a \text{ are separated by a multiple of } \pi$$

$$= -\frac{2}{\pi} \left[ \frac{1}{(m-n)^2} - \frac{1}{(m+n)^2} \right] \text{ with odd}$$

$$\int_0^{\pi} \delta(x-x') \phi(x') dx' = \lambda \phi(x)$$

$$= \int_0^{\pi} \delta(x-x') \phi'(x') dx' = \phi'(x)$$

$$+ [\delta(x-x') \phi(x')]_0^{\pi} = \delta(x-\pi) \phi(\pi) - \delta(x) \phi(0)$$

$$\int_{-\pi/2}^{\pi} \delta(x-\pi) dx + \int_{\pi}^{\pi+2} \delta(x-\pi) dx = 1$$

$$+ \int_0^{\pi} \delta(x) dx = 1$$

$$\lambda \phi(x) = \phi'(x) + \delta(x-\pi) \phi(\pi) - \delta(x) \phi(0)$$

$$\phi(x) = e^{\lambda x} + \alpha \delta(x-\pi) + \beta \delta(x)$$

$$\int_0^{\pi} \delta(x-x') f(x') dx' = f(x) - \int_{-\pi/2}^0 \delta(x-x') f(x') dx' - \int_{\pi}^{\pi+2} \delta(x-x') f(x') dx'$$

$$\int_0^{\pi} \delta(x-x') dx' = \int_{a-\epsilon}^a \delta(y) dy$$

$f(x)$  a continuous fn.

$$\int_{-\pi/2}^{\pi/2} f(x') \delta(x-x') dx' = f'(x) \quad x \neq 0$$

Suppose  $f(x, x')$  made periodic in  $x$  and  $x'$

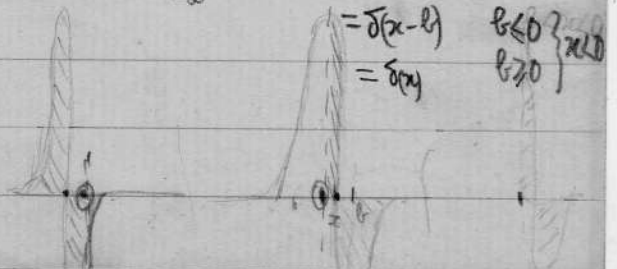
$$f(x, x') = f(x+\pi, x') = f(x, x'+\pi)$$

$$\int_{-\infty}^{\infty} f(x, x') dx' = -\delta(x-0)$$

$$\int_{-\infty}^{\pi} f(0, x') dx' = \infty \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\int_{-\infty}^{\pi} f(x, x') dx' = \delta(x-0) - \delta(x) \quad \begin{cases} x < 0 \\ x > 0 \end{cases}$$

$$= \delta(x-0) \quad \begin{cases} x < 0 \\ x > 0 \end{cases}$$



$$\phi\left(\frac{\pi}{2}\right) = \phi\left(-\frac{\pi}{2}\right)$$

$$\int_{-\infty}^x \mu(x, x') dx' = \delta(x-x') - \delta(x) \quad x > 0$$

$$= \delta(x-x') \quad x' \leq 0 \quad x < 0$$

$$= \delta(x) \quad x' \geq 0 \quad x > 0$$

$$= P(x, x')$$

$$\int_{-\pi/2}^{\pi/2} \mu(x, x') \phi(x') dx' = \left[ \phi(x') P(x, x') \right]_{-\pi/2}^{\pi/2}$$

$$- \int_{-\pi/2}^{\pi/2} \phi'(x') P(x, x') dx' = \phi\left(\frac{\pi}{2}\right) \left\{ \begin{array}{l} x > 0 \\ x < 0 \end{array} \right\} \left\{ \begin{array}{l} \delta(x - \frac{\pi}{2}) - \delta(x) \\ \delta(x) - \delta(x + \frac{\pi}{2}) \end{array} \right\}$$

spurious

$$\lim_{\epsilon \rightarrow 0} = \phi\left(\frac{\pi}{2}\right) \left\{ \begin{array}{l} \delta(x - \frac{\pi}{2}) - \delta(x) \\ \delta(x) - \delta(x + \frac{\pi}{2}) \end{array} \right\} \quad x \geq 0$$

$$= \mp \phi\left(\frac{\pi}{2}\right) \delta(x) \quad x \leq 0$$

$$2^{\text{nd}} \lim = \int_0^{\pi/2} \phi'(x') \left\{ \frac{\delta(x-x')}{\delta(x)} \right\} dx' + \int_{-\pi/2}^0 \phi'(x') \left\{ \frac{\delta(x)}{\delta(x-x')} \right\} dx'$$

$$= \left\{ \begin{array}{l} \phi'(x) - \int_{-\epsilon}^0 \phi'(x') \delta(x-x') dx' + \delta(x) [\phi(0) - \phi(\frac{\pi}{2})] \\ \delta(x) [\phi(\frac{\pi}{2}) - \phi(0)] + \phi'(x) - \int_0^{\epsilon} \phi'(x') \delta(x-x') dx' \end{array} \right\}$$

$$\lim_{\epsilon \rightarrow 0} + \phi'(x) = \left\{ \begin{array}{l} -\delta(x) \phi(0) + \int_{-\epsilon}^0 \phi'(x') \delta(x-x') dx' \\ \delta(x) \phi(0) + \int_0^{\epsilon} \phi'(x') \delta(x-x') dx' \end{array} \right\}$$

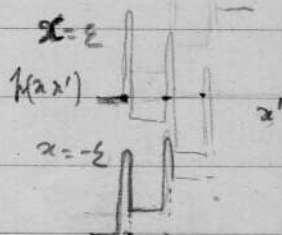
If  $\phi$  is continuous, then is  $\mp \delta(x) \phi(0) + \left\{ \begin{array}{l} \int_{-\epsilon}^0 \\ \int_0^{\epsilon} \end{array} \right\} \phi'(x') \delta(x-x') dx'$

$$\Rightarrow \mp \delta(x) \phi(0) + \left\{ \begin{array}{l} \int_x^{x+\epsilon} \delta(y) \phi'(x+y) dy \\ \int_{x-\epsilon}^x \delta(y) \phi'(x+y) dy \end{array} \right\}$$

$$x - x' = y$$

$$\delta_1(x) = \delta(x) \quad x < 0$$

$$= -\delta(x) \quad x > 0$$

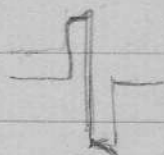


$$P(x, x') = \delta(x) \quad x' \leq 0 \quad x \geq 0$$

$$= \delta(x-x') \quad x' \geq 0 \quad x < 0$$

$$= \delta(x-x') \quad x' \leq 0 \quad x \leq 0$$

$$= \delta(x) \quad x' \geq 0 \quad x > 0$$



$$d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu$$

$$(\mu, \nu = 0, 1, 2, 3)$$

$$(i, k = 1, 2, 3)$$

$$\bar{x}^0 = x^0 + \psi_0(x_1, x_2, x_3)$$

$$\bar{\gamma}_{\mu\nu} = \gamma_{\rho\sigma} \frac{\partial x^\rho}{\partial \bar{x}^\mu} \frac{\partial x^\sigma}{\partial \bar{x}^\nu}$$

$$\bar{x}^i = \psi_i(x_1, x_2, x_3)$$

$$\gamma_{\rho\sigma} = \bar{\gamma}_{\mu\nu} \frac{\partial \bar{x}^\mu}{\partial x^\rho} \frac{\partial \bar{x}^\nu}{\partial x^\sigma} = \bar{\gamma}_{\mu\nu} \left\{ \delta_\rho^0 \delta_\sigma^0 + \frac{\partial \psi^\mu}{\partial x^\rho} \delta_\sigma^0 + \frac{\partial \psi^\nu}{\partial x^\sigma} \delta_\rho^0 \right\}$$

$$\bar{x}^\mu = \delta_0^\mu x^0 + \psi^\mu(x_1, x_2, x_3)$$

$$\gamma_{00} = \bar{\gamma}_{00}$$

$$\gamma_{0k} = \bar{\gamma}_{0\mu} \frac{\partial \psi^\mu}{\partial x^k}$$

$$\gamma_{k0} = \bar{\gamma}_{\mu 0} \frac{\partial \psi^\mu}{\partial x^k}$$

$$\gamma_{ik} = \bar{\gamma}_{\mu\nu} \frac{\partial \psi^\mu}{\partial x^i} \frac{\partial \psi^\nu}{\partial x^k}$$

$$d\bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} dx^\nu = \left\{ \delta_0^\mu \delta_\nu^0 + \frac{\partial \psi^\mu}{\partial x^\nu} \right\} dx^\nu$$

$$d\bar{x}^0 = dx^0 + \frac{\partial \psi^0}{\partial x^k} dx^k$$

$$d\bar{x}^i = \frac{\partial \psi^i}{\partial x^k} dx^k$$

$$d\bar{x}^0 + \frac{\bar{\gamma}_{0i}}{\bar{\gamma}_{00}} d\bar{x}^i = dx^0 + \frac{\partial \psi^0}{\partial x^k} dx^k + \frac{\bar{\gamma}_{0i}}{\bar{\gamma}_{00}} \frac{\partial \psi^i}{\partial x^k} dx^k = dx^0 + \frac{\gamma_{0k}}{\gamma_{00}} dx^k = d\theta$$

$$\bar{\gamma}_{0i} \frac{\partial \psi^i}{\partial x^k} = \bar{\gamma}_{0\mu} \frac{\partial \psi^\mu}{\partial x^k} - \bar{\gamma}_{00} \frac{\partial \psi^0}{\partial x^k} = \gamma_{0k} - \bar{\gamma}_{00} \frac{\partial \psi^0}{\partial x^k}$$

$$\delta \int \sqrt{-g} dx^0 dx^1 dx^2 dx^3 = - \int (G^{\mu\nu} - \frac{1}{2} g^{\mu\nu} G) \delta g_{\mu\nu} \sqrt{-g} dx^0 dx^1 dx^2 dx^3$$





$$L = L_0 - \frac{e}{c} \dot{\mathbf{r}} \cdot (\mathbf{A}_1 \mathbf{i} + \mathbf{A}_2 \mathbf{j} + \mathbf{A}_3 \mathbf{k})$$

$$\mathbf{A} = \frac{1}{2} [\mathbf{H}, \mathbf{r}]$$

$$\dot{p}_x = \frac{\partial L}{\partial x} = m\dot{x} - \frac{e}{c} \dot{\mathbf{r}} \cdot \mathbf{A}_x$$

$$\dot{p}_x = -\frac{\partial H^*}{\partial x} \quad \dot{x} = \frac{\partial H^*}{\partial p_x}$$

Ad. charge

$$\dot{p}_x = -\frac{\partial H^*}{\partial x} - \frac{e}{c} \dot{\mathbf{r}} \cdot \mathbf{A}_x$$

$$H^* = H_0^* + \frac{e}{2mc} \mathbf{H} \cdot \mathbf{r}$$

Electric field given by

$$\text{curl } \mathbf{E} = -\frac{1}{c} \dot{\mathbf{H}}$$

$$\mathbf{H} = \text{curl } \mathbf{A}$$

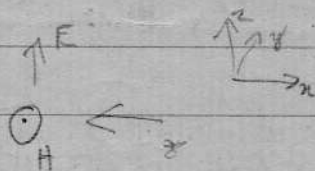
$$\mathbf{E} = -\frac{1}{c} \dot{\mathbf{A}} - \text{grad } \phi$$

$\phi$  of order  $A$

This results in  $m\dot{x}$  being increased by  $+\frac{e}{c} \dot{\mathbf{r}} \cdot \mathbf{A}_x + e \frac{\partial \phi}{\partial x}$

$$\dot{p}_x \text{ becomes } -\frac{\partial H^*}{\partial x} + e \frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x} (H^* + e\phi)$$

$$\dot{x} = +\frac{\partial}{\partial p_x} (H^* + e\phi)$$



$$\frac{\partial H_0}{\partial x} = \frac{\partial E_z}{c \partial t}$$

$$\begin{aligned} \kappa_1 &= \alpha \cos(l_1 x_1 - ct) v \\ \kappa_2 &= \alpha_2 \cos(l_1 x_1 - ct) v \end{aligned}$$

$$\alpha_1 l_1 - \alpha_2 v^2 = 0$$

$$\begin{aligned} \kappa_p &= \alpha_p \cos(l_1 x_1 vt) e^{i v c t} \\ \kappa_u &= \alpha_u \cos(l_1 x_1 vt) e^{i v c t} \\ \alpha_1 l_1 - \alpha_u &= 0 \end{aligned}$$

$$E_2 = \frac{\partial \kappa_1}{\partial t} - \frac{\partial \kappa_u}{\partial x_2} = v \sin(l_1 x_1 - ct) v \{ \alpha_2 - l_2 \alpha_u \}$$

$$E_2 = \frac{\partial \kappa_2}{\partial t} - \frac{\partial \kappa_u}{\partial x_2} = \alpha_2$$

$$\kappa_p = \alpha_p \sin(l_1 x_1 vt) \sin(l_2 x_2 vt) \sin(l_3 x_3 vt) \sin ct v$$

$$E_2 = \frac{\partial \kappa_1}{\partial t} - \frac{\partial \kappa_u}{\partial x_2} = v \sin(l_1 x_1 vt) \sin(l_3 x_3 vt) \{ \alpha_2 \sin(l_2 x_2 vt) \cos ct v - \alpha_{l_2} \cos(l_2 x_2 vt) \sin ct v \}$$

$$\begin{aligned} &= e^{i v c t} \\ &= e^{i v c t} \\ &= e^{i v c t} \\ &= e^{i v c t} \end{aligned}$$



$$\left[ \frac{1}{\sqrt{2}} \right]$$

$$r_n = r$$

$$r^2 \frac{\partial^2 \psi}{\partial r^2} + r \frac{\partial \psi}{\partial r} - r^2 \psi$$

$$\psi = J_0(i n r)$$

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{n+2m}}{m! (n+m)!}$$

$$Y_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m}}{(m!)^2} \left\{ \log \frac{1}{z} - \frac{\partial}{\partial z} T(v, m+1) \right\}$$

$$f(x) = i(\sin \theta + \cos \theta) +$$

As the quantum configuration may be referred with a constant, the quantum then goes out of existence.

② Different terms in solution  $\psi/\psi = 2 \cos \psi$  or  $\psi/\psi = 2 \cos \psi$  refer to different states of atoms + molecules. Indeed of some total energy.

\* If  $x$  and the  $J$ 's are real and if  $\bar{x}_\alpha$  denotes the conjugate imaginary of  $x_\alpha$ , then by equating the conjugate imaginaries of both sides of (8) we get

$$x = \sum_\alpha e^{-i(\alpha\omega t)} \bar{x}_\alpha(J) = \sum_\alpha \bar{x}_\alpha(J + \alpha h) e^{-i(\alpha\omega t)}$$

Comparing this with eq (8) we find that

$$\bar{x}_\alpha(J + \alpha h) = x_{-\alpha}(J)$$

This relation is brought out more clearly if we change the notation. For  $x_\alpha(J)$  write  $x(J, J-\alpha h)$ .

Then 
$$\bar{x}(J+\alpha h, J) = x(J, J+\alpha h)$$

which shows that there is some kind of symmetry in the way in which  $x(J, J+\alpha h)$  is related to the two sets of variables to which it explicitly refers. Our expansion for  $x$  is now

$$x = \sum_\alpha x(J, J-\alpha h) e^{i(\alpha\omega t)} = \sum_\alpha e^{i(\alpha\omega t)} x(J+\alpha h, J)$$

If  $y$  can also be expanded into  $y = \sum_\beta y(J, J-\beta h) e^{i(\beta\omega t)}$

then 
$$xy = \sum_{\alpha, \beta} x(J, J-\alpha h) e^{i(\alpha\omega t)} y(J, J-\beta h) e^{i(\beta\omega t)} \\ = \sum_{\alpha, \beta} x(J, J-\alpha h) y(J-\alpha h, J-\alpha h-\beta h) e^{i(\alpha+\beta, \omega t)}$$

By again using (6),

or, the amplitudes of  $xy$  are given by

$$xy(J, J-\gamma h) = \sum_\alpha x(J, J-\alpha h) y(J-\alpha h, J-\gamma h) \quad (11)$$

These formulae provide a way of representing  $q$ -numbers by  $c$ -numbers. Suppose that in the expressions  $x(J, J-\alpha h)$ , considered merely as a functions of the  $J$ 's, we <sup>substitute</sup> for each  $J$  the  $c$ -number  $n+h$ , and denote the resulting  $c$ -numbers by  $x(n, n-\alpha)$ ,  $y(n, n-\beta)$ . We take a series of values all differing from each other by  $a$  and  $b$ . We consider the aggregate of all the  $c$ -numbers  $x(n, n-\alpha)$  <sup>expressed as</sup>  $e^{i\omega(n, n-\alpha)t}$ , where the  $n$ 's represent the values of the  $q$ -number  $x$  for all values of the  $q$ -numbers  $J$ .

Eq. (10) shows that  $x(n, n-\alpha) = i\omega(n, n-\alpha)$  while eq. (8) gives

$$xy(n, n-\gamma) = \sum_\alpha x(n, n-\alpha) y(n-\alpha, n-\gamma)$$

which is just Heisenberg's law of multiplication. By means of this law one can obtain the  $c$ -numbers which represent any function of the  $p$ 's and  $q$ 's that has the necessary periodic properties <sup>when one knows the  $c$ -numbers which represent the  $p$ 's and  $q$ 's</sup>. And representation is thus complete in itself.

$$W(J=2h) < W(J)$$

such as (the set of zero's say)

Suppose all the  $x(J, J=2h)$  ( $x \neq 0$ ) vanish when some set of  $c$ -numbers is substituted for the  $J$ 's. These  $c$ -numbers would then define a metastable state of the atom. If in addition each  $x(J, J=2h)$  vanishes when  $J=2h$  is put equal to any <sup>any</sup> <sup>negative</sup> integral multiple of  $h$ , the state would be the normal state.

In this case every  $x(n, m)$  would vanish if an  $n$  or  $m$  is negative, provided we take the  $n$ 's and  $m$ 's all integers. It appears that we should have to take the  $n$ 's and  $m$ 's all integers in our representations in order that the resulting  $c$ -numbers may be of physical importance.

uses 3's & 4's

$$b = e^{-i(H'-W)t/h} e^{i(H-W)t'/h}$$

$$e^{-iWt/h} + e^{iWt'/h} = t + t'$$

$$b + b^{-1} = t'$$

$$bWb^{-1} = W + W' - H'$$

$$b = e^{i(H-W)t/h} e^{-i(H'-W')t'/h} e^{i(H'-W')t/h}$$

$$\frac{dx}{dt} = [x, H-W]$$

if  $x$  is fn of undashed variables only

$$\frac{dx'}{dt'} = [x', H'-W']$$

if  $x'$  - dashed -

$H$  is a fn only of undashed variables and dashed variables that commute with  $H'-W'$ .

$H'$  is a fn only of dashed variables and undashed variables that commute with  $H-W$ .

$$b + b^{-1} = t'$$

$$b(H-W)b^{-1} = H'-W'$$

$$b_1 b_2 b_1 = b_2 b_1 b_2$$

$$b_2^{-1} b_1 b_2 = b_1 b_2 b_1^{-1}$$

$$b_1^{-1} b_2$$

$$b_1 b_2 = b_3$$

$$b_3^{-1} = b_2 b_3 b_2$$

$$b_2 b^{-1} \text{ commutes with } H-W$$

$$b^{-1} x' b = H'-W'$$

$$b = e^{i(H'-H)t/h}$$

$$[t+t', H'-W']t - (H-W)t' = -t' + t$$

$$[t-t', \quad ] =$$

$$[t, (H'-W')t - (H-W)t'] = -t'$$

$$[t', (H'-W')t - (H-W)t'] = t$$

$$H-W = \text{fn.}(x, x' b^{-1})$$

$$b(H-W)b^{-1} = H'-W' = \text{fn.}(b_2 b^{-1}, x')$$

$$b + b^{-1} = t'$$

$$b = e^{iWt}$$

$$\frac{dx}{dt} = [x, H-W]$$

$$\frac{dx'}{dt'} = [x', H'-W']$$

any X

$$X^2 = x$$

$$eXb^{-1} = -X$$

Any no that commutes with  $x$  either commutes with  $X$  or is a fn of the single variable  $b$ , or is a fn of  $b$  and numbers that commute with  $X$ .

Suppose  $\xi$  is a fn of  $b$  and of nos that commute with  $X$ .

$$XrX^{-1} = X' r X'^{-1}$$

$$X^2 = X'^2$$

Put

$P$ , of course, commutes with  $k_1, c_1$  and  $c_2$ . We have

$$c_1 c_2 = \frac{1}{4} m^2 e^4 \left( \frac{1}{k_1^2} - \frac{1}{p^2} \right) = \frac{1}{4} m^2 e^4 \frac{\epsilon_1^2}{k_1^2} \quad (27)$$

$$c_2 c_1 = \frac{1}{4} m^2 e^4 \left( \frac{1}{k_2^2} - \frac{1}{p^2} \right) = \frac{1}{4} m^2 e^4 \frac{\epsilon_2^2}{k_2^2} \quad (28)$$

$$\epsilon_1 = \sqrt{1 - \frac{k_1^2}{p^2}}, \quad \epsilon_2 = \sqrt{1 - \frac{k_2^2}{p^2}}$$

The eccentricities  $\epsilon_1$  and  $\epsilon_2$  are constants, and commute with  $P$  and  $k$  and with each other. Put

$$c_1 = \frac{1}{2} m^2 \epsilon_1 / k_1 e^{-ix}$$

$X$  is a constant and so commutes with  $c_1$ .

Since  $k$  commutes with  $c_1 e^{ix}$  and with  $\epsilon_1 / k_1$ , it must commute with  $e^{-ix} e^{ix}$ , i.e.

$$k e^{-ix} e^{ix} = e^{-ix} e^{ix} k = e^{-ix} (k - k_1) e^{ix}$$

$$k e^{ix} = e^{-ix} (k + k_1) e^{ix}$$

Thus for the interchange of  $x$  and  $k$  shows that

$X$  is canonically conjugate to  $k$ . We now have

$X$  corresponds to the classical theory to the angle between the major axis of the ellipse and the line  $O-C$ .

$$c_1 = \frac{1}{2} m^2 \epsilon_1 / k_1 e^{-ix} = \frac{1}{2} m^2 \epsilon_1 / k_1 e^{-ix}$$

and from (27) and (28)

$$c_2 = \frac{1}{2} m^2 e^{ix} \epsilon_1 / k_1 = \frac{1}{2} m^2 \epsilon_2 / k_2 e^{ix}$$

The expression for  $1/r$  thus takes the form

$$\frac{1}{r} = \frac{m^2}{k_1 k_2} \left( 1 + \frac{1}{2} \frac{k_2 e^{ix}}{k_1} e^{-ix} + \frac{1}{2} \frac{k_1}{k_2} \epsilon_2 e^{ix} e^{-ix} \right) \quad (29)$$

$$J = -n$$

$$i \hbar \frac{\partial}{\partial t}$$

$$x = J^{\frac{1}{2}} e^{i\omega t} + e^{-i\omega t} J^{\frac{1}{2}}$$

$$[x, J] = i \{ J^{\frac{1}{2}} e^{i\omega t} - e^{-i\omega t} J^{\frac{1}{2}} \}$$

$$f(x) = x^{\frac{1}{2}} e^{2\pi i n x^{\frac{1}{2}}}$$

$n$  an arb. integer

is ind. of  $n$  only when  $x^{\frac{1}{2}}$  is a positive integer

$$n = J$$

so that  $f(x)$  ind. of  $n$  is nec. for  $f(x)$  to be a fn of  $x$  only, i.e. to commute with  $x$ .

fn of a fn is a fn.

Algebraic equation may be used for definition or part definition of a fn. at the start

We do not define all the properties of the  $q$ -numbers in use, at the start but add to them later as required.

$$[S, J_r] = \frac{1}{2} \sum_k \{ \hbar_k [q_k, J_r] + [q_k, J_r] \hbar_k \}$$

$$\frac{\partial S}{\partial \hbar_k} = \sum_r \hbar_r \frac{\partial q_k}{\partial \hbar_r}$$

(c)

$$[S, q_r] = \frac{1}{2} \sum_k \{ \omega_k [J_k, q_r] + [J_k, q_r] \omega_k \}$$

$$\frac{\partial S}{\partial \omega_k} = \sum_r \omega_r \frac{\partial J_k}{\partial \omega_r}$$

$$[S, [q_i, J_j]] = \frac{1}{2} \sum_k \{ \hbar_k [q_k [q_i, J_j]] + [q_k [q_i, J_j]] \hbar_k + \omega_k [J_k [q_i, J_j]] + [J_k [q_i, J_j]] \omega_k \}$$

$$= -[q_i, [J_j, S]] + [J_j, [q_i, S]]$$

$$= \frac{1}{2} [q_i, \sum_k \{ \hbar_k [q_k, J_j] + [q_k, J_j] \hbar_k \}] + \frac{1}{2} [J_j, \sum_k \{ \omega_k [J_k, q_i] + [J_k, q_i] \omega_k \}]$$

$$= \frac{1}{2} \sum_k \{ \hbar_k [q_i [q_k, J_j] + 2[q_i, \hbar_k] [q_k, J_j] + [q_i, [q_k, J_j]] \hbar_k - \omega_k [J_j [J_k, q_i] + 2[J_j, \omega_k] [J_k, q_i] + [J_j, [J_k, q_i]] \}$$

$$= \frac{1}{2} \sum_k \{ \hbar_k [q_k [q_i, J_j] + [q_k [q_i, J_j]] \hbar_k + \omega_k [J_k [J_j, q_i] + [J_k [J_j, q_i]] \omega_k \} + [q_i, J_j] + [J_j, q_i]$$

$$a \cdot f(\theta) = \sum_n \frac{(i\hbar)^n}{n!} f^{(n)}(\theta) a_n$$

$$a e^{i\theta} = \sum_n \frac{(i\hbar)^n}{n!} e^{i\theta} a_n$$

$$e^{i\theta} a = \sum_n \frac{\hbar^n}{n!} a_n e^{i\theta}$$

$$e^{i\theta} a e^{-i\theta} = \sum_n \frac{\hbar^n}{n!} a_n$$

$$\theta = S/\hbar$$

$$e^{iS/\hbar} a e^{-iS/\hbar} = \sum_n \frac{1}{n!} a_{SSS...S}$$

$$\alpha = \alpha(H, q)$$

$$e^{2\pi i S/\hbar} q e^{-2\pi i S/\hbar}$$

$$e^{iS/\hbar} \hbar e^{-iS/\hbar} = f(H) = J$$

$$H = e^{iS/\hbar} f(H) e^{-iS/\hbar}$$

$$e^{2\pi i S/\hbar} \hbar e^{-2\pi i S/\hbar} = \hbar + 2\pi$$

$$e^{2\pi i S/\hbar} e^{-iS/\hbar} e^{-2\pi i S/\hbar}$$

$$e^{iS/\hbar} = K e^{iS/\hbar} \text{ where } K \text{ commutes with the } \hbar$$

$$e^{-2\pi i n S/\hbar} K \text{ commutes with the } \hbar$$

$$a_n = \frac{1}{n!} [a, \theta]^n \theta - \theta^n$$

$$a_n = a_{SSS...S}$$

$$e dS = \hbar e dq_k + \frac{1}{2} d\hbar$$

$$= \hbar e dq_k - \hbar e d\bar{q}_k + d(\hbar e q_k)$$

$$d(\hbar e q_k - \hbar e \bar{q}_k) = \Delta \hbar e dq_k + \hbar e d\Delta q_k$$

$$d(\hbar e q_k - \hbar e \bar{q}_k - \hbar e \Delta q_k) = \Delta \hbar e dq_k$$

$$d(\hbar e q_k - \hbar e \bar{q}_k) = + \Delta \hbar e dq_k - d\hbar e d\bar{q}_k$$

$$= 2 = \sum_n \frac{1}{n!} q_{2\pi i, 2\pi i, \dots} (0)$$

$$e^{iS/\hbar} = c e^{2\pi i (S/\hbar)} e^{iS/\hbar}$$

$$[f(S), q_r] = \frac{1}{2} \sum_k \{ \phi(S) [J_k + f(S), q_r] + [J_k + f(S), q_r] \phi(S) \}$$

$S$  is  $S$  with  $\hbar$  replaced by  $f(S)$

$$\hbar = e^{-iS/\hbar} J e^{iS/\hbar} = e^{-iS/\hbar} J e^{iS/\hbar}$$

$$q_r = e^{-iS/\hbar} \hbar e^{iS/\hbar} = e^{-iS/\hbar} (\hbar + 2\pi i) e^{iS/\hbar} = e^{-iS/\hbar} \hbar e^{iS/\hbar} + e^{-iS/\hbar} 2\pi i e^{iS/\hbar}$$

$$e^{-2\pi i (nS)/\hbar}$$

$K$  commutes with the  $\omega$ 's and  $J$ 's, and is a  $c$ -number

|    |   |   |   |   |   |
|----|---|---|---|---|---|
| 10 | 7 | 3 | 4 | 7 | 3 |
| 4  | 1 | 3 | 2 | 1 | 3 |
| 2  | 1 | 1 | 0 | 1 | 1 |

25

algebra. It is a particular case of a more general axiom, which may be enunciated thus:— any number that commutes with  $e$

The paper deals entirely (except for which is pure mathematics) of contributions to the new quantum mechanics of the atom that was introduced by Heisenberg in 1925. The <sup>main</sup> essential point of H's theory is that the dynamical variables of the classical theory are to be replaced by matrices (as far as possible) satisfying <sup>formally</sup> the same equations as the classical variables. The equations of the classical theory  $xy = yx$ , which express the law of multiplication, <sup>commute</sup> are no longer in general true for matrices and the equation that replace in paper 1 the hypothesis is just found that this equation is to be replaced (on the Q.T) by  $xy - yx = \frac{h}{2\pi} [x, y]$  where  $h$  is Planck's constant and the expression  $[x, y]$  is the analogue of the P.B. of the classically on the classical theory. The justification of this assumption is the similarity in the algebraic laws governing the expression  $xy - yx$  and the P.B.  $[x, y]$  of the classical theory.

In paper 2 it is attempted to establish the theory from a new point of view in which pre-eminence is given to the fact that the dynamical variables in the quantum theory do not obey the common laws of multiplication, and the fact that matrices can be found to represent the variables is considered of less importance. A theory of inferring variables is <sup>given</sup> obtained which is applied to the Hydrogen atom, the Balmer formula being obtained, and also, in paper 3, to a number of problems relating to the orbital motion of electron in an atom, ~~such~~ such as the anomalous Zeeman effect and the ratio of the intensities of the spectral lines in a multiplet. In paper 4 the <sup>theoretical</sup> ~~theoretical~~ after being modified so as to give results in agreement with the principle of relativity, is applied to the scattering of radiation by a free electron. Compton's formula for the frequency of the scattered radiations is obtained, and also a new formula for the intensity of the scattered radiation, which agrees with experiment. The same question is treated in paper 4 by a different method. Paper 5 is entirely pure mathematics, and the axioms <sup>there made</sup> are not suitable for the later form of the quantum theory (see paper 8).

The whole work is based on the idea of a quantum mechanics which was put forward by H in 2 f.p. --  
The papers after the 5<sup>th</sup> have been greatly influenced by Schrödinger's notions *Ann d. Phys* and are based on his <sup>ideas</sup> ~~methods~~.

The individual results in the paper have in part been obtained independently by other investigators at about the same time e.g. the quantum condition given in paper 1 was obtained independently by B & J, the Balmer formula <sup>deduced in paper 2</sup> was obtained ind. by Pauli.

The main original points in the paper are as follows:

<sup>introduced in paper 1 and used in 2, 3, 4</sup>  
The relation between  $x-y$  on the (R.T) and the P.B. (Erg) of the classical theory.

The method of infomising variables given in paper 2 and its application in papers 2 & 3.

The theory of the Compton effect given in paper 4, which leads to a formula for the intensity of the scattered radiation in agreement with experiment.

In places we have ~~used~~ <sup>drawn</sup> on the work of other investigators, in Q.M. to which reference is given in footnotes. <sup>chief cases of this are that papers 6-10 are largely based on S's ideas, and that the theory of paper 8 is a development of Heisenberg's theory of fluctuations given in 2 f. Phys.</sup>

The papers contain original contributions to the theory of Q.M. introduced by Heisenberg in 2 f. Phys., which work of Heisenberg provided the fundamental ideas on which the whole of the paper are based. The main original points of the papers are as follows: -

The quantum conditions of paper 1.

The theory of infomising variables of paper 2 and its applications in papers 2 & 3, 4.

The theory of the Compton effect in paper 4.

The theory of statistical theory of paper 7 § 3, 4. (The statistics which are required by Pauli's exclusion principle had been previously worked out, without reference to Q.M., by Fermi, 2 f. Phys. -)

The new formulation of quantum mechanics in paper 8.

The radiation theory of paper 9 and its application to dispersion in paper 10.

The perturbation theory of paper 9 § 6.

Some of these results have been obtained <sup>at about the same time</sup> independently by other investigators, namely, the quantum conditions by Born and Jordan, the statistical theory of paper 7 (in part) by Heisenberg, and a theory mathematically equivalent to that of paper 8 has been given by Jordan.

algebra. It is a particular case of a more general axiom which may be enunciated thus:— If a number commutes with every number that commutes with the  $q$ -numbers  $q, q_1, q_2, \dots, q_n$  and with one number that does not commute with  $q$ , but commutes with  $q_1, q_2, \dots, q_n$ , then it commutes with every number that commutes with  $q_1, q_2, \dots, q_n$ . (Note that a number  $x$  that commutes with every number must be a  $c$ -number, on account of the <sup>assumption</sup> axiom that if  $x$  is a  $q$ -number, there is



In paper 9 <sup>on the basis of paper 8</sup> it is shown that the problem is indeed of the interaction of a number of light quanta with an atom. It is shown that, owing to the special statistics <sup>obeyed</sup> satisfied by light quanta, the Hamiltonian <sup>function</sup> that describes the interaction is of the same form as that which describes the interaction of the atom with quantized electromagnetic waves. This enables one to build up a self-consistent theory of radiation which contains the essential features of both the corpuscular and wave theories. This theory completes that of paper 755 by accounting in a natural manner for spontaneous emission. In paper 10 this theory is applied to dispersion.

are based on S's ideas papers

The later papers (6-10) have been greatly influenced by S's work.

which may provide a powerful mathematical method of obtaining the matrices of H's theory although the physical ideas <sup>of</sup> statistical in these papers, which involve practically a return to classical mechanics, has ~~not~~ proved to be attainable. An example of this method is given in paper 6, where the Compton effect is treated and the results are obtained in agreement with those of paper 14. In paper 7 it is shown that two kinds of statistics are allowed by Q.M. and discussed (2.3.4) and also in 3.5 a quantum perturbation theory is given which is shown to account correctly for the absorption and stimulated emission of radiation by an atom, although not the spontaneous emission.

In paper 8 a <sup>and more general</sup> new basis is provided for Q.M. the whole theory which supersedes the theory of paper 2. It is shown that the <sup>canonical</sup> operators of the quantum theory ~~and the~~ can be represented by matrices in a number of different matrix schemes, not only in one as in H's ~~first~~ theory. The ~~operator~~ variables are to be regarded as completely defined when one is given the matrices that represent them in any one scheme, and one can ~~then determine~~ <sup>then</sup> it being possible to obtain the matrices that represent them in any <sup>other</sup> scheme by simple transformation laws. The theory provides a general method for the physical interpretation of Q.M. It appears that it is permissible to talk about dynamical variables having specified numerical values provided that one does not give numerical values simultaneously to two variables that do not commute, e.g. one can state that for a given atomic ~~system~~ system  $x = a$  and  $y = b$  where  $x$  and  $y$  are dynamical variables (coordinates or velocities <sup>etc.</sup> of particles) and  $a$  and  $b$  are numbers, provided  $x$  and  $y$  <sup>have</sup> ~~are~~ not the same. If  $x$  and  $y$  are the same, then for a system in which  $x = a$  one can only say that there is a certain probability of  $y$  having any specified value. This probability is given by the theory, and is <sup>given by the theory</sup> ~~the~~ the diagonal element  $(a, a)$  of the matrix  $\rho$  in a matrix scheme in which  $x$  is a diagonal matrix. Then if one is given, say, the initial values of certain of the dynamical variables, one can calculate the prob. on any variable having any specified value at any subsequent time. From this <sup>given</sup> theory all the special assumptions contained in previous forms of the Q.T. can be deduced as special cases.