PROPERTIES

OF

EXPANDING UNIVERSES

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List of Contents

Abstract

Some implications and consequences of the expansion of the universe are examined. In Chapter 1 it is shown that this expansion creates grave difficulties for the Hoyle–Narlikar theory of gravitation.

Chapter 2 deals with perturbations of an expanding homogeneous and isotropic universe. The conclusion is reached that galaxies cannot be formed as a result of the growth of perturbations that were initially small. The propagation and absorption of gravitational radiation is also investigated in this approximation.

In Chapter 3 gravitational radiation in an expanding universe is examined by a method of asymptotic expansions. The 'peeling–off' behaviour and the asymptotic group are derived.

Chapter 4 deals with the occurrence of singularities in cosmological models. It is shown that a singularity is inevitable provided that certain very general conditions are satisfied.

Introduction

The idea that the universe is expanding is of recent origin. All the early cosmologies were essentially stationary and even Einstein, whose theory of relativity is the basis for almost all modern developments in cosmology, found it natural to suggest a static model of the universe.

However, there is a very grave difficulty associated with a static model such as Einstein's, which is supposed to have existed for an infinite time. For, if the stars had been radiating energy at their present rates for an infinite time, they would have needed an infinite supply of energy. Further, the flux of radiation now would be infinite.

Alternatively, if they had only a limited supply of energy, the whole universe would by now have reached thermal equilibrium, which is certainly not the case. This difficulty was noticed by Olbers, who however was not able to suggest any solution. The discovery of the recession of the nebulae by Hubble led to the abandonment of static models in favour of ones which were expanding.

Clearly there are several possibilities: the universe may have expanded from a highly dense state a finite time ago (the so-called "big bang" model); another is that the present expansion may have been preceded by a contraction which, in Its turn may have been preceded by another expansion (the "bouncing" or oscillating model); however, this model suffers from the same difficulties over entropy as the static model. Finally, it is possible that the expansion may have been proceeding at much the same rate for an infinite time. It is then necessary to postulate some form of continual creation of matter in order to prevent the expansion from reducing the density to zero. This leads to the "steady-state" model which, although expanding, presents the same appearance at all times.

The early cosmologies naturally placed man at or near the centre of the universe, but since the time of Copernicus we have been demoted to a medium-sized planet going round a medium-sized star somewhere near the edge of a fairly average galaxy. We are now so humble that we would not claim to occupy any special position. However, observations seem to indicate that, within experimental error (which is fairly high), galaxies have a spatially isotropic distribution around us. As we are

not claiming any special position, the distribution must be isotropic about every point. This implies that the distribution must be spatially homogeneous as well as isotropic. Of course, this homogeneity and isotropy hold only on a large scale; locally there are considerable departures from both.
Robertson and Walker have shown that the metric of a model that is spatially homogeneous and isotropic may be written in the form:

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right], \qquad \kappa = 0, \pm 1$$

This form will be used extensively in the following chapters. In Chapter 1 it will be shown that the Hoyle–Narlikar theory of gravitation is incompatible with a metric of this form. In Chapter 2 perturbations of this form will be considered in a linearized approximation and, in Chapter 3, gravitational radiation will be considered in a model which tends asymptotically to this form.

Certain of the Robertson–Walker models possess "horizons". There are two types: particle horizons and event horizons. A particle horizon is said to exist when an observer's past light cone does not intersect the world line of every particle in the universe (or extended world line in the case of a particle which has been created). An example of a model with a particle horizon is the Einstein–de Sitter model which has

$$\kappa = 0, \qquad R = t^{\frac{2}{3}}\,.$$

This is a "big–bang" model as indeed are all the Robertson–Walker models that satisfy the Einstein equations:

$$R_{ab} - \tfrac{1}{2}g_{ab}R = T_{ab}$$

...and contain matter whose pressure is greater than minus one–third the density. An event horizon exists when there are events that a given observer will never see. The steady–state model $\left(\kappa = 0,\ R = e^t\right)$ is an example of one with an event horizon. Horizons will be further discussed in Chapter 4 which also deals with the occurrence of singularities of space–time and their connection with topology.

Each chapter is self–contained and has its own references. The following notation is used throughout: space–time is taken to be a Riemannian manifold with metric tensor g_{ij}. This is taken to have signature -2 except in Chapter 2 where, in order to facilitate comparison with previous work, the signature is $+2$. Covariant differentiation is indicated by a semi–colon. Units are employed in which c, the speed of light, and k, the gravitational constant, equal one.

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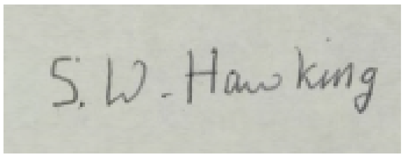
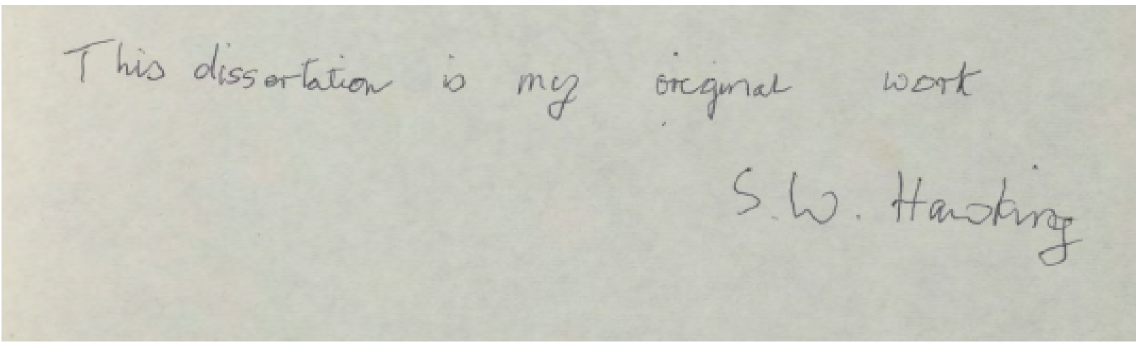


S. W. Hawking



Chapter 1
The Hoyle–Narlikar Theory of Gravitation

1. Introduction

The success of Maxwell's equations has led to electrodynamics being normally formulated in terms of fields that have degrees of freedom independent of the particles in them. However, Gauss suggested that an action–at–a–distance theory in which the action travelled at a finite velocity might be possible. This idea was developed by Wheeler and Feynman [**2**, **3**] who derived their theory from an action principle that involved only direct interactions between pairs of particles. A feature of this theory was that the 'pseudo'–fields introduced are the half–retarded plus half–advanced fields calculated from the world–lines of the particles.

However, Wheeler and Feynman, and, in a different way, Hogarth (3), were able to show that, provided certain cosmological conditions were satisfied, these fields could combine to give the observed field. Hoyle and Narlikar [**4**] extended the theory to general space–times and obtained similar theories for their 'C'–field [**5**] and for the gravitational field [**6**]. It is with these theories that this chapter is concerned.

It will be shown that in an expanding universe the advanced fields are infinite, and the retarded fields finite. This is because, unlike electric charges, all masses have the same sign.

2. The Boundary Condition

Hoyle and Narlikar derive their theory from the action:

$$A = \sum_{a \neq b} \sum \iint G(a, b)\, da\, db,$$

where the integration is over the world-lines of particles a, b... In this expression G is a Green function that satisfies the wave equation:

$$G(x,x')_{;ij}\, g^{ij} + \tfrac{1}{6} R\, G(x,x') = \frac{\delta^4(x,x')}{\sqrt{-g}}\,.$$

where S is the determinant of $g_{ij} Sincethedoublesumintheaction A issymmetricalbetweenallpairlrso$

$$A = \sum_{a\neq b}\sum \iint G^*(a,b)\, da\, db$$

where $\quad G^*(a,b) = \tfrac{1}{2}G(a,b) + \tfrac{1}{2}G(b,a).$

Thus $\quad G^* \quad$ must be the time-symmetric Green function, and can be written:

$$G^* = \tfrac{1}{2}G_{\text{ret}} + \tfrac{1}{2}G_{\text{adv}} \qquad \text{where } G_{\text{ret}}$$

and $G_{\text{adv}} aretheretardedandadvancedGreenfunctions. Byrequiringthattheactionbestationaryunder$
equations :

$$\left[\sum_{a\neq b}\sum \tfrac{1}{6} m^{(a)}(x)\, m^{(b)}(x)\right]\left(R_{ik} - \tfrac{1}{2} g_{ik} R\right)$$

$$= -T_{ik} + \sum_{a\neq b}\sum \tfrac{1}{3} m^{(a)}\left[g_{ik} m^{(b)r}{}_{;r} - m^{(b)}_{;ik}\right] + 2\left(m^{(a)}_{il}\, m^{(b)l}_{j}\right) - \tfrac{1}{4} g_{ik}\, m^{(a);r}\, m^{(b)}_{j;r}$$

where $\quad m^{(a)}(x) = \int G^*(x,a)\, da.$

However, as a consequence of the particular choice of Green function, the contraction of the field–equations is satisfied identically. There are thus only 9 equations for the 10 components of g_{ij} and the system is indeterminate.

Hoyle and Narlikar therefore impose $\sum m^{(a)} = m_0 = \text{const.}$, as the tenth equation. By then making the 'smooth–fluid' approximation, that is by putting

$$\sum_{a\neq b}\sum m^{(a)} m^{(b)} \approx m_0^2,$$

they obtain the Einstein field–equations:

$$\tfrac{1}{6} m_0^2 \left(R_{ik} - \tfrac{1}{2} R g_{ik}\right) = -T_{ik}.$$

There is an important difference, however, between these field–equations in the direct–particle interaction theory and in the usual general theory of relativity. In the general theory of relativity, any metric that satisfies the the field–equations is admissible, but in the direct–particle interaction theory only those solutions of the field–equations are admissible that satisfy the additional requirement:

$$m_0(x) = \sum m^{(a)}(x) = \sum \int G^*(x,a)\,da = \tfrac{1}{2}\sum \int G_{\text{ret}}(x,a)\,da + \tfrac{1}{2}\sum \int G_{\text{adv}}(x,a)\,da$$

This requirement is highly restrictive; it will be shown that it is not satisfied for the cosmological solutions of the Einstein field–equations, and it appears that it cannot be satisfied for any models of the universe that either contain an infinite amount of matter or undergo infinite expansion.

The difficulty is similar to that occurring in Newtonian theory when it is recognized that the universe might be infinite.

The Newtonian potential ϕ obeys the equation:

$$\Box\phi = -\kappa\rho \qquad (\rho > 0),$$

where ρ is the density. In an infinite static universe, ϕ would be infinite, since the source always has the same sign. The difficulty was resolved when it was realized that the universe was expanding, since in an expanding universe the retarded solution of the above equation is finite by a sort of 'red–shift' effect. The advanced solution will be infinite by a 'blue–shift' effect. This is unimportant in Newtonian theory, since one is free to choose the solution of the equation and so may ignore the infinite advanced solution and take simply the finite retarded solution.

Similarly in the direct–particle interaction theory the m field satisfies the equation:

$$\Box m - \tfrac{1}{6}Rm = \mathcal{N} \qquad (\mathcal{N} > 0),$$

where $\mathcal{N}$ is the density of world–lines of particles. As in the Newtonian case, one may expect that the effect of the expansion of the universe will be to make the retarded solution finite and the advanced solution infinite. However, one is now not free to choose the finite retarded solution, for the equation is derived from a direct–particle interaction action–principle symmetric between pairs of particles, and one must choose for m half the sum of the retarded and advanced solutions. We would expect this to be infinite, and this is shown to be so in the next section.

3. The Cosmological Solutions

The Robertson–Walker cosmological metrics have the form

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\big(d\theta^2 + \sin^2\theta\, d\phi^2\big)\right].$$

Since they are conformally flat, one can choose coordinates in which they become

$$ds^2 = \Omega^2\big[d\tau^2 - d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)\big] = \Omega^2 \eta_{ab}\, dx^a dx^b,$$

where η_{ab} is the flat–space metric tensor and

$$\Omega = \Omega(\tau,\rho) = \frac{R(t)}{\sqrt{[t+\frac{1}{4}\kappa(\tau+\rho)^2][t+\frac{1}{4}\kappa(\tau-\rho)^2]}}. \tag{7}$$

For example, for the Einstein–de Sitter universe

$$\kappa = 0, \quad R(t) = \left(\tfrac{t}{T}\right)^{2/3}, \qquad (0 < t < \infty),$$

$$\Omega = R\cdot\left(\tfrac{\tau}{T}\right)^{2/3}, \qquad (0 < \tau < \infty),$$

$$\rho = \tau^{2/3}\,\xi^{1/3}.$$

For the steady–state (de Sitter) universe

$$\kappa = 0, \quad R(t) = e^{t/T}, \qquad (-\infty < t < \infty).$$

$$\Omega = R\cdot\frac{1}{\tau}, \qquad (-\infty < t < \infty),$$

$$r = \rho\left(\frac{\tau}{T} - Te^{t/T}\right).$$

The Green function $G^*(a,b)$ obeys the equation

$$\Box G^*(a,b) - \tfrac{1}{6}R\,G^*(a,b) = \frac{\delta^4(a,b)}{\sqrt{-g}}.$$

From this it follows that

$$\frac{1}{\Omega^2}\frac{\partial}{\partial x^a}\left(\Omega^2\eta^{ab}\frac{\partial}{\partial x^b}G^*\right) = \frac{1}{\Omega^2}\frac{\partial}{\partial x^a}\left(\eta^{ab}\frac{\partial}{\partial x^b}S\right) = \Omega^{-3}\,\delta^4(a,b).$$

If we let $G^* = \Omega^{-1}S$, then

$$\Omega^2 \frac{\partial}{\partial x^a}\left(\eta^{ab}\frac{\partial}{\partial x^b} S\right) = \delta^4(a,b).$$

This is simply the flat–space Green function equation, and hence

$$G^*(t_1,0;t_2,\rho) = \frac{\Omega(t_1)}{8\pi\,\Omega(t_2)\,\rho}\Big[\delta(\rho-(t_2-t_1)) + \delta(\rho+(t_2-t_1))\Big].$$

The m–field is given by

$$m(t,x) = \int G^* \mathcal{N}\sqrt{-g}\,dx = \tfrac{1}{2}(m_{\rm ret}+m_{\rm adv}).$$

For universes without creation (e.g. the Einstein–de Sitter universe),

$$\mathcal{N} = R^3 n, \qquad n = \text{const}.$$

For Universes with creation (steady state) $\mathcal{N} = n, \quad n = \text{const}.$

$$M_{\rm adv}(\tau_1) = \Omega^{-1}(\tau_1)\int \frac{\mathcal{N}\,\Omega^3(\tau_2)}{4\pi r}\,4\pi r^2\,d\tau_2,$$

where the integration is over the future light cone. This will normally be infinite in an expanding universe, e.g. in the Einstein–de Sitter universe:

$$M_{\rm adv}(\tau_1) = \left(\frac{\tau_1}{T}\right)^{-2}\int_{\tau_1}^{\infty} n\left(\tfrac{\tau_2}{T}\right)^2\,d\tau_2 = \infty.$$

In the steady–state universe

$$M_{\rm adv}(\tau_1) = \left(\tfrac{-T}{\tau_1}\right)^{-1}\int_{\tau_1}^{0} n\left(\tfrac{-T}{\tau_2}\right)^3(\tau_2-\tau_1)\,d\tau_2 = \infty.$$

By contrast, on the other hand, we have

$$M_{\rm ret}(\tau_1) = \Omega^{-1}(\tau_1)\int \frac{\mathcal{N}\,\Omega^3}{4\pi r}\,4\pi r^2\,dr,$$

where the integration is over the past light cone. This will Normally be finite, e.g. in the Einstein–de Sitter universe

$$M_{\rm ret}(\tau_1) = \left(\frac{\tau_1}{T}\right)^{-2}\int_0^{\tau_1} n\left(\tfrac{\tau_2}{T}\right)^2(\tau_2-\tau_1)\,d\tau_2 = \tfrac{1}{2}nT^2.$$

While in the steady–state universe

$$M_{\rm ret}(\tau_1) = \left(\tfrac{-T}{\tau_1}\right)^{-1} \int_{-\infty}^{\tau_1} n \left(\tfrac{-T}{\tau_2}\right)^3 (\tau_2 - \tau_1)\, d\tau_2 = \tfrac{1}{2} n T^2.$$

Thus it can be seen that the solution $m =$ const. of the equation

$$\Box m + \tfrac{1}{6} R m = \mathcal{N}$$

is not, in a cosmological metric, the half–advanced plus half–retarded solution since this would be infinite. In fact, in the case of the Einstein–de Sitter and steady–state metrics, it is the pure retarded solution.

4. The 'C'–Field

Hoyle and Narlikar derive their direct–particle interaction theory of the 'C'–field from the action

$$A = \sum_{a \neq b} \iint \hat{G}(a,b)\, i_a\, k_b\, da^4 db^4.$$

where the suffixes a, b refer to differentiation of $\hat{G}(a,b)$ on the world–lines of a, b respectively. $\hat{G}$ is a Green function obeying the equation

$$\Box \hat{G}(x,x') = \frac{\delta^4(x,x')}{\sqrt{-g}}.$$

We define the 'C'–field by

$$C(x) = \sum \hat{G}(x,a)\, i_a\, da^i,$$

and the matter–current J^κ by

$$J^\kappa(y) = \sum \int \delta^4(y,b)\, db^\kappa.$$

Then

$$C(x) = \int \hat{G}(x,y)\, J^\kappa(y)_{;\kappa} \sqrt{-g}\, dx',$$

so that

$$\Box C = J^\kappa{}_{;\kappa}.$$

We thus see that the sources of the 'C'–field are the places where matter is created or destroyed.

As in the case of the m–field, the Green function must be time–symmetric, that is

$$\hat{G}(a,b) = \tfrac{1}{2}\,\hat{G}_{\text{ret}}(a,b) + \tfrac{1}{2}\,\hat{G}_{\text{adv}}(a,b)$$

Hoyle and Narlikar claim that if the action of the 'C'–field is included along with the action of the 'm'–field, a universe will be obtained that approximates to the steady–state universe on a large scale although there may be local irregularities. In this universe, the value of C will be finite and its gradient time–like and of unit magnitude.

Given this universe, we may check it for consistency by calculating the advanced and retarded 'C'–fields and finding if their sum is finite. We shall not do this directly but will show that the advanced field is infinite while the retarded field is finite.

Consider a region in space–time bounded by a three–dimensional space–like hypersurface $\mathcal{D}$ at the present time, and the past light cone Σ of some point P to the future of $\mathcal{D}$.

By Gauss's theorem

$$\int_V \Box C\,\sqrt{-g}\,dx^4 = \int_{\Sigma+\mathcal{D}} \frac{\partial C}{\partial n}\,dS = \int J^{\kappa}{}_{;\kappa}\,\sqrt{-g}\,dx^4.$$

Let the advanced field produced by sources within V be C'. Then C' and $\frac{\partial C'}{\partial n}$ will be zero on Σ, and hence

$$\int_V J^{\kappa}{}_{;\kappa}\,\sqrt{-g}\,dx^4 = \int_{\mathcal{D}} \frac{\partial C'}{\partial n}\,dS.$$

But $J^{\kappa}{}_{;\kappa}$ is the rate of creation of matter $= n$ (const.) in the steady–state universe, and hence

$$\int_{\mathcal{D}} \frac{\partial C'}{\partial n}\,dS = nV.$$

As the point P is taken further into the future, the volume of the region V tends to infinity. However, the area of the hypersurface $\mathcal{D}$ tends to a finite limit owing to horizon effects. Therefore the gradient $\frac{\partial C'}{\partial n}$ must be infinite. A similar calculation shows the gradient of the retarded field to be finite. Their sums cannot therefore give the field of unit gradient required by the Hoyle–Narlikar theory.

It is worth noting that this result was obtained without assumptions of a smooth distribution of matter or of conformal flatness.

5. Conclusion

It is one of the weaknesses of the Einstein theory of relativity that although it furnishes field equations it does not provide boundary conditions for them. Thus it does not give a unique model for the universe but allows a whole series of models. Clearly a theory that provided boundary conditions and thus restricted the possible solutions would be very attractive. The Hoyle–Narlikar theory does just that (the requirement that $m = \frac{1}{2}m_{\mathrm{ret}} + \frac{1}{2}m_{\mathrm{adv}}$ is equivalent to a boundary condition). Unfortunately, as we have seen above, this condition excludes those models that seem to correspond to the actual universe, namely the Robertson–Walker models.

The calculations given above have considered the universe as being filled with a uniform distribution of matter. This is legitimate if we are able to make the 'smooth–fluid' approximation to obtain the Einstein equations. Alternatively, if this approximation is invalid, it cannot be said that the theory yields the Einstein equations.

It might possibly be that local irregularities could make m_{adv} finite, but this has certainly not been demonstrated and seems unlikely in view of the fact that, in the Hoyle–Narlikar direct–particle interaction theory of their 'C'–field,

which is derived from a very similar action–principle, it can be shown without assuming a smooth distribution that the advanced 'C' field will be infinite in an expanding universe with creation.

The reason that it is possible to formulate a direct–particle interaction theory of electrodynamics that does not encounter this difficulty of having the advanced solution infinite is that in electrodynamics there are equal numbers of sources of positive and negative sign. Their fields can cancel each other out and the total field can be zero apart from local irregularities. This suggests that a possible way to save the Hoyle–Narlikar theory would be to allow masses of both positive and negative sign. The action would be

$$A = \sum_{a \neq b} q_a q_b \iint G^*(a,b)\, da\, db, \qquad (q_a, q_b = \pm 1),$$

where q_a, q_b are gravitational charges analogous to electric charges. Particles of positive q in a positive 'm'–field and particles of negative q in a negative 'm'–field would have the normal gravitational properties, that is, they would have positive gravitational and inertial masses.

A particle of negative q in a positive 'm'–field would still follow a geodesic. Therefore it would be attracted by a particle of positive q. Its own gravitational effect however would be to repel all other particles. Thus it would have the properties of the negative mass described by Bondi (8), that is, negative gravitational mass and negative inertial mass.

Since there does not seem to be any matter having these properties in our region of space (where $m = \text{const.} > 0$) there must clearly be separation on a very large scale. It would not be possible to identify particles of negative q with antimatter, since it is known that antimatter has positive inertial mass. However, the introduction of negative masses would probably raise more difficulties than it would solve.

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Chapter 2
Perturbations

1. Introduction

Perturbations of a spatially isotropic and homogeneous expanding universe have been investigated in a Newtonian approximation by Bonnor [**Bonnor1957**] and relativistically by Lifshitz [**Lifshitz1946**], Lifshitz and Khalatnikov [**LifshitzKhalatnikov1963**], and Irvine [**Irvine1965**]. Their method was to consider small variations of the metric tensor. This has the disadvantage that the metric tensor is not a physically significant quantity, since one cannot directly measure it, but only its second derivatives. It is thus not obvious what the physical interpretation of a given perturbation of the metric is. Indeed it need have no physical significance at all, but merely correspond to a coordinate transformation. Instead it seems preferable to deal in terms of perturbations of the physically significant quantity, the curvature.

2. Notation

Space–time is represented as a four–dimensional Riemannian space with metric tensor g_{ab} of signature +2. Covariant differentiation in this space is indicated by a semi–colon. Square brackets around indices indicate antisymmetrisation and round brackets symmetrisation. The conventions for the Riemann and Ricci tensors are:

$$\nabla_{a;[bc]} = 2\, R^{p}{}_{acb}\, \nabla_{p},$$

$$R_{ab} = R^{p}{}_{apb}.$$

η_{abcd} is the alternating tensor.

Units are such that κ, the gravitational constant, and c, the speed of light, are one

3. The Field Equations

We assume the Einstein equations:

$$R_{ab} - \tfrac{1}{2} g_{ab} R = -T_{ab}$$

where T_{ab} is the energy–momentum tensor of matter. We will assume that the matter consists of a perfect fluid. Then,

$$T_{ab} = \mu\, u_a u_b + p\, h_{ab}$$

where u_a is the velocity of the fluid, $u_a u^a = -1$:

$$\mu \text{ is the density,} \qquad p \text{ is the pressure,}$$

$h_{ab} = g_{ab} + u_a u_b$ is the projection operator into the hyperplane orthogonal to u_a,

$$h_{ab} u^b = 0.$$

We decompose the gradient of the velocity vector u_a as

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3} h_{ab}\, \theta - \dot{u}_a u_b$$

where

$$\dot{u}_a = u_{a;b} u^b \quad \text{is the acceleration,}$$

$$\theta = u^a{}_{;a} \quad \text{is the expansion,}$$

$$\sigma_{ab} = u_{(c;d)} h^c{}_a h^d{}_b - \tfrac{1}{3} h_{ab} \theta \quad \text{is the shear,}$$

$$\omega_{ab} = u_{[c;d]} h^c{}_a h^d{}_b \quad \text{is the rotation of the flow lines } u^a.$$

We define the rotation vector ω_a as

$$\omega_a = \tfrac{1}{2} \eta_{abcd}\, \omega^{cd} u^b.$$

We may decompose the Riemann tensor R_{abcd} into the Ricci tensor R_{ab} and the Weyl tensor C_{abcd}:

$$R_{abcd} = C_{abcd} - g_{a[c} R_{d]b} + g_{b[c} R_{d]a} - \tfrac{1}{3} R g_{a[c} g_{d]b},$$

$$C_{abcd} = C_{[ab][cd]}, \qquad C^a{}_{bca} = 0 = C_{a[bcd]}.$$

C_{abcd} is that part of the curvature that is not determined locally by the matter. It may thus be taken as representing the free gravitational field (Jordan, Ehlers and Kundt[**JordanKundt1960**]). We may decompose it into its "electric" and "magnetic" components:

$$E_{ab} = -C_{abpq}\, u^p u^q,$$

$$H_{ab} = -\tfrac{1}{2} C_{apqr}\, \eta^{qr}{}_{bs}\, u^p u^s,$$

$$C_{ab}{}^{cd} = 8u_{[a} E_{b]}{}^{[c} u^{d]} + 4\delta_{[a}{}^{[c} E_{b]}{}^{d]} - 2\eta_{ab}{}^{cd} u^p H_p{}^{[c} u^{d]} - 2\eta^{cdrs} u_r H_{s[a} u_{b]}.$$

$$E_{ab} = E_{(ab)}, \qquad H_{ab} = H_{(ab)},$$

$$E_a{}^a = H_a{}^a = 0,$$

$$E_{ab} u^b = H_{ab} u^b = 0.$$

E_{ab} and H_{ab} each have five independent components.

We regard the Bianchi identities,

$$R_{ab[cd;e]} = 0,$$

as field equations for the free gravitational field.

Then

$$C_{abcd}{}^{d} = -R_{c[b;a]} + \tfrac{1}{6} g_{c[b} R_{;a]},$$

(Kundt and Trümper[**KundtTrumper1962**]).

Using the decompositions given above, we may write these in a form analogous to the Maxwell equations:

$$h^b{}_e H_{bc;d} h^{cf} + 3H_{ab}\omega^b - \eta_{abcd} u^b \sigma^c{}_e H^{de} = \tfrac{3}{2} h_a{}^b \mu_{;b}, \tag{1}$$

$$h^b{}_e E_{bc;d} h^{cf} + 3E_{ab}\omega^b - \eta_{abcd} u^b \sigma^c{}_e E^{de} = (\mu + p)\,\omega_a, \tag{2}$$

$$\begin{aligned}\perp \dot{E}_{ab} + h^c{}_{(a}\eta_{b)cde} u^c H^{de} + E_{ab}\theta - E^c{}_{(a}\omega_{b)c} - E^c{}_{(a}\sigma_{b)c} - \eta_{cde(a}\eta_{bpqr} u^c u^p \sigma^d{}_q E^{er} \\ + 2H^c{}_{(a}\eta_{b)cde} u^c \dot{u}^e = -\tfrac{1}{2}(\mu + p)\sigma_{ab},\end{aligned} \tag{3}$$

$$\begin{aligned}\perp \dot{H}_{ab} - h^c{}_{(a}\eta_{b)cde} u^c E^{de} + H_{ab}\theta - H^c{}_{(a}\omega_{b)c} - H^c{}_{(a}\sigma_{b)c} - \eta_{cde(a}\eta_{bpqr} u^c u^p \sigma^d{}_q H^{er} \\ + 2H^c{}_{(a}\eta_{b)cde} u^c \dot{u}^e = 0.\end{aligned} \tag{4}$$

where $\perp$ indicates projection by h_{ab} orthogonal to u_a (c.f. Trümper (7)).

The contracted Bianchi identities give,

$$(R_{ab} - \tfrac{1}{2}g_{ab}R)^{;b} = -T_{ab}{}^{;b} = 0,$$

$$\dot{\mu} + (\mu + p)\theta = 0, \tag{5}$$

$$(\mu + p)\dot{u}_a + p_{,b}h^b{}_a = 0. \tag{6}$$

The definition of the Riemann tensor is,

$$u_{a;[bc]} = 2R_{apbc}u^p.$$

Using the decompositions as above we may obtain what may be regarded as "equations of motion".

$$\dot{\theta} = 2\omega^2 - 2\sigma^2 - \tfrac{1}{3}\theta^2 + \dot{u}^a{}_{;a} - \tfrac{1}{2}(\mu + 3p), \tag{7}$$

$$\perp \dot{\omega}_{ab} = -\tfrac{2}{3}\omega_{ab}\theta + 2\sigma_{c[a}\omega_{b]}{}^c + \dot{u}_{[p;q]}h^p{}_a h^q{}_b, \tag{8}$$

$$\begin{aligned}\perp \dot{\sigma}_{ab} = {} & E_{ab} - \omega_{ac}\omega^c{}_b - \sigma_{ac}\sigma^c{}_b - \tfrac{2}{3}\sigma_{ab}\theta \\ & - \tfrac{1}{3}h_{ab}(2\omega^2 - 2\sigma^2 + \dot{u}^c{}_{;c}) + \dot{u}_a\dot{u}_b + \dot{u}_{(p;q)}h^p{}_a h^q{}_b,\end{aligned} \tag{9}$$

where $2\omega^2 = \omega_{ab}\omega^{ab}$, $\quad 2\sigma^2 = \sigma_{ab}\sigma^{ab}$.

We also obtain what may be regarded as equations of constraint:

$$\theta_{,b}h^{ba} = \tfrac{3}{2}\Big[(\omega_{bc;b} + \sigma^c{}_{b;c})h^{ca} - \dot{u}^b(\omega_{ab} + \sigma_{ab})\Big], \tag{10}$$

$$\omega^a{}_{;a} = 2\omega_a\dot{u}^a, \tag{11}$$

$$H_{ab} = -h^c{}_{(a}\eta_{b)cde}u^c\left(\omega^{d;e} + \sigma^{d;e}\right). \tag{12}$$

We consider perturbations of a universe that in the undisturbed state is conformally flat, that is

$$C_{abcd} = 0.$$

By equations (1)–(3), this implies

$$\sigma_{ab} = \omega_{ab} = 0, \qquad h^b{}_a\mu_{;b} = 0, \qquad \theta_{;b}h^{ba}.$$

If we assume an equation of state of the form, $\mu = \mu(\rho)$, then by (6), (10),

$$\mu_{,b}h^b{}_a = 0 = \dot{\mu}u_a.$$

This implies that the universe is spatially homogeneous and isotropic since there is no direction defined in the 3-space orthogonal to u_a.

In this universe we consider small perturbations of the motion of the fluid and of the Weyl tensor. We neglect products of small quantities and perform derivatives with respect to the undisturbed metric. Since all the quantities we are interested in with the exception of the scalars, $\mu, \dot{\mu}, \theta$ have unperturbed value zero, we avoid perturbations that merely represent coordinate transformation and have no physical significance.

To the first order the equations (1)–(4) and (7)–(9) are

$$E_{ab}{}^{;b} = \tfrac{1}{3} h_a{}^b \mu_{,b}, \tag{13}$$

$$H_{ab}{}^{;b} = (\mu + h)\,\omega_a, \tag{14}$$

$$\dot{E}_{ab} + E_{c(a}\theta + h^c{}_{(a}\eta_{b)de}u^d H^e{}_c{}^{;c} = -\tfrac{1}{2}(\mu + h)\,\sigma_{ab}, \tag{15}$$

$$\dot{H}_{ab} + H_{c(a}\theta - h^c{}_{(a}\eta_{b)de}u^d E^e{}_c{}^{;c} = 0, \tag{16}$$

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 + \dot{u}^a{}_{;a} - \tfrac{1}{2}(\mu + 3h), \tag{17}$$

$$\dot{\omega}_{ab} = -\tfrac{2}{3}\omega_{ab}\theta + \dot{u}_{[a;p]}h^p{}_b h^q{}_a, \tag{18}$$

$$\dot{\sigma}_{ab} = E_{ab} - \tfrac{2}{3}\sigma_{ab}\theta - \tfrac{1}{3}h_{ab}\dot{u}^c{}_{;c} + \dot{u}_{(p;q)}h^p{}_a h^q{}_b. \tag{19}$$

From these we see that perturbations of rotation or of E_{ab} or H_{ab} do not produce perturbations of the expansion or the density. Nor do perturbations of E_{ab} and H_{ab} produce rotational perturbations.

4. The Undisturbed Metric

Since in the unperturbed state the rotation and acceleration are zero, u_a must be hypersurface orthogonal.

$$u_a = \tau_{,a},$$

where τ measures the proper time along the world lines. As the surfaces $\tau =$ constant are homogeneous and isotropic they must be 3-surfaces of constant curvature. Therefore the metric can be written,

$$ds^2 = -d\tau^2 + \Omega^2 d\gamma^2,$$

where

$$\Omega = \Omega(\tau),$$

$d\gamma^2$ is the line element of a space of zero or unit positive or negative curvature.

We define t by

$$\frac{dt}{d\tau} = \frac{1}{\Omega},$$

then

$$ds^2 = \Omega^2 \left(-dt^2 + d\gamma^2\right).$$

In this metric,

$$u_a = (-\Omega, 0, 0, 0),$$

$$\therefore \quad \theta = \frac{3\dot{\Omega}}{\Omega} = \frac{3\Omega'}{\Omega^2}$$

(prime denotes differentiation with respect to t).

Then, by (5), (7)

$$\dot{\mu} = -(\mu + h)\,\frac{3\dot{\Omega}}{\Omega}, \tag{20}$$

$$3\frac{\ddot{\Omega}}{\Omega} = -\tfrac{1}{2}(\mu + 3h). \tag{21}$$

If we know the relation between μ and h, we may determine Ω. We will consider the two extreme cases, $h = 0$ (dust) and $h = \frac{1}{3}\mu$ (radiation). Any physical situation should lie between these.

For $h = 0$

By (20), $\quad \mu = \dfrac{M}{\Omega^3}, \quad M = \text{const.}$

$$\therefore \quad \frac{3}{M}\,\Omega\ddot{\Omega} - \tfrac{1}{2}\Omega^2 = 0,$$

$$\therefore \quad \frac{3}{M}\,\dot{\Omega}^2 - \frac{1}{\Omega} = E, \qquad E = \text{const.}$$

(a) For $E>0$,

$$\Omega = \tfrac{1}{2E}\Big(\cosh\sqrt{\tfrac{EM}{3}}\,t - 1\Big), \qquad \tau = \tfrac{1}{2E}\Big(\sqrt{\tfrac{3}{EM}}\,\sinh\sqrt{\tfrac{EM}{3}}\,t + t\Big);$$

(b) For $E=0$,

$$\Omega = \frac{M}{12}\,t^2, \qquad \tau = \frac{M}{36}\,t^3;$$

(c) For $E<0$,

$$\Omega = -\tfrac{1}{2E}\Big(1-\cos\sqrt{-\tfrac{EM}{3}}\,t\Big), \qquad \tau = -\tfrac{1}{2E}\Big(t-\sqrt{\tfrac{3}{-EM}}\,\sin\sqrt{-\tfrac{EM}{3}}\,t\Big).$$

E represents the energy (kinetic + potential) per unit mass. If it is non–negative the universe will expand indefinitely, otherwise it will eventually contract again. By the Gauss–Codazzi equations ${}^*R$, the curvature of the hypersurface $\tau=$ const. is

$$ {}^*R = 2\left(-\tfrac{1}{3}\theta^2+\mu\right) = -\frac{2EM}{\Omega^2}.$$

If $E>0$, $\quad {}^*R = -\dfrac{6}{\Omega^2}, \qquad M=\dfrac{3}{E};$

$$E=0, \quad {}^*R=0;$$

If $E<0$, $\quad {}^*R = \dfrac{6}{\Omega^2}, \qquad M=-\dfrac{3}{E}.$

For $h=\frac{1}{3}\mu$

$$\dot\mu = -4\frac{\dot\Omega}{\Omega},$$

$$3\frac{\ddot\Omega}{\Omega} = -\mu, \qquad \mu = \frac{M}{\Omega^4},$$

$$\therefore \quad \frac{3}{M}\dot\Omega^2 - \frac{1}{\Omega^2} = E.$$

(a) For $E>0$:

$$\Omega = \frac{1}{E}\sinh t, \qquad \tau = \frac{1}{E}(\cosh t - 1), \qquad {}^*R = -\frac{6}{\Omega^2};$$

(b) For $E=0$:

$$\Omega = t, \qquad \tau = \tfrac{1}{2}t^2, \qquad {}^*R = 0;$$

(c) For $E < 0$:

$$\Omega = \frac{1}{E}\sin t, \qquad \tau = \frac{1}{E}(\cos t - 1), \qquad {}^*R = \frac{6}{\Omega^2}.$$

5. Rotational Perturbations

By (6)

$$\dot{u}_{[c;d]}h^c{}_a h^d{}_b = -\frac{\omega_{ab}\,\dot{h}}{\mu + h}.$$

$$\dot{\omega}_{ab} = -\omega_{ab}\left(\tfrac{2}{3}\theta + \tfrac{\dot{h}}{\mu+h}\right).$$

For $h = 0$,

$$\omega = \frac{\omega_0}{\Omega^2}.$$

For $h = \frac{\mu}{3}$,

$$\dot{\omega} = -\omega\left(\tfrac{2}{3}\theta + \tfrac{1}{4}\frac{\dot{\mu}}{\mu}\right) = -\tfrac{5}{3}\omega\theta,$$

$$\therefore \quad \omega = \frac{\omega_0}{\Omega^5}.$$

Thus rotation dies away as the universe expands. This is in fact a statement of the conservation of angular momentum in an expanding universe.

6. Perturbations of Density

For $h = 0$ we have the equations,

$$\dot{\mu} = -\mu\theta,$$

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 - \tfrac{1}{2}\mu.$$

These involve no spatial derivatives. Thus the behaviour of one region is unaffected by the behaviour of another. Perturbations will consist in some regions having slightly higher or lower values of θ than the average. If the universe as whole has a value of E greater than zero, a small perturbation will still have E greater than zero and will continue to expand. It will not contract to form a galaxy. If the universe has a value of E less than zero, a small perturbation can contract. However it will only begin contracting at a time $\delta\tau$ earlier than the whole universe begins contracting, where

$$\frac{\delta\tau}{\tau_c} = \frac{\delta E}{E_0},$$

τ_c is the time at which the whole universe begins contracting. There is only any real instability when $E = 0$. This case is of measure zero relative to all the possible values E can have. However this cannot really be used as an argument to dismiss it as there might be some reason why the universe should have $E = 0$.

For a region with energy $-\delta E$, in a universe with $E = 0$,

$$\Omega = \frac{1}{4\delta E}\left(t^2 - \frac{t^4}{12} + \cdots\right),$$

$$\tau = \frac{1}{12\delta E}\left(t^3 - \frac{t^5}{20} + \cdots\right),$$

$$\mu = \frac{3}{6E\,\Omega^2} = \frac{4}{3}\tau^{-2}\left(1 + \frac{(\delta E)^{2/3}}{2^{1/3}}\tau^{2/3} + \cdots\right).$$

For $E = 0$, $\quad \mu = \frac{4}{3}\tau^{-2}$.

Thus the perturbation grows only as $\tau^{2/3}$. This is not fast enough to produce galaxies from statistical fluctuations even if these could occur. However, since an evolutionary universe has a particle horizon (Rindler (8), Penrose (9)) different parts do not communicate in the early stages. This makes it even more difficult for statistical fluctuations to occur over a region until light had time to cross the region.

For $h = \frac{\mu}{3}$

$$\dot\mu = -\tfrac{4}{3}\mu\theta,$$

$$\dot\theta = -\tfrac{1}{3}\theta^2 - \mu + \dot u^a{}_{;a},$$

$$\dot u_a = -\frac{h^b{}_a\,\mu_{,b}}{4\mu}.$$

As before, a perturbation cannot contract unless it has a negative value of E. The action of the pressure forces makes it still more difficult for it to contract. Eliminating θ,

$$\mu\ddot\mu - \tfrac{5}{4}\dot\mu^2 - \tfrac{5}{3}\mu^3 + \tfrac{4}{3}\mu^2\dot u^a{}_{;a} = 0,$$

$$\ddot u^a = \dot u^a{}_{;b}\,h^b{}_a{}^b + \dot u^a\dot u^a = -\tfrac{1}{4}\,h^{ac}\frac{(h^b{}_a\mu_{,b})_{;c}}{\mu} \quad \text{to our approximation.}$$

$$h^{ac}\nabla_c h^b{}_a \nabla_b$$

is the Laplacian in the hypersurface $\tau =$ constant.

We represent the perturbation as a sum of eigenfunctions $S^{(n)}$ of this operator, where

$$S^{(n)}{}_{;a} u^a = 0,$$

$$h^{ac}(h^b{}_a S^{(n)}{}_{,b})_{;c} = -\frac{n^2}{\Omega^2} S^{(n)}.$$

These eigenfunctions will be hyperspherical and pseudohyperspherical harmonics in cases (c) and (a) respectively and plane waves in case (b). In case (c) n will take only discrete values but in (a) and (b) it will take all positive values.

$$\mu = \mu_0 \left(1 + \sum_n B^{(n)} S^{(n)}\right),$$

where μ_0 is the undisturbed density.

$$\ddot{B}^{(n)}\mu_0 - \tfrac{1}{2}\dot{B}^{(n)}\mu_0 - B^{(n)}\left(\tfrac{4}{3}\mu_0^2 - \tfrac{n^2}{3\Omega^2}\mu_0\right) = 0.$$

As long as $\mu_0 > \frac{n^2}{4\Omega^2}$, $B^{(n)}$ will grow.

For $\mu_0 \gg \frac{n^2}{4\Omega^2}$,

$$B^{(n)} \simeq C\,\tau + D\,\tau^{-1}.$$

These perturbations grow for as long as light has not had time to travel a significant distance compared to the scale of the perturbation ($\sim \frac{\Omega}{n}$). Until that time pressure forces cannot act to even out perturbations.

When $\frac{n^2}{\Omega^2} \gg \mu_0$,

$$\ddot{B}^{(n)} + \dot{B}^{(n)}\frac{\dot{\Omega}}{\Omega} + \tfrac{n^2}{3}B^{(n)} = 0,$$

$$\therefore \quad B^{(n)} \simeq C\,\Omega^{-\frac{1}{2}} e^{\pm i\frac{n}{\sqrt{3}}t}.$$

We obtain sound waves whose amplitude decreases with time. These results confirm those obtained by Lifshitz and Khalatnikov (3).

From the foregoing we see that galaxies cannot form as the result of the growth of small perturbations. We may expect that other non– gravitational forces will have an effect smaller than pressure equal

7. The Steady-state Universe

To obtain the steady-state universe we must add extra terms to the energy–momentum tensor. Hoyle and Narlikar (10) use,

$$T_{ab} = \mu\, u_a u_b + h_{ab} - C_a C_b + \tfrac{1}{2} g_{ab} C_c C^d. \tag{20}$$

where,

$$C_a = C_{;a}, \qquad C_{;a}{}^{a} = -\dot{j}_a{}^{a}, \qquad j_a = -(\mu + h) u_a.$$

Since $T_{ab}{}^{;b} = 0$,

$$\dot{\mu} + (\mu + h)\theta + u^a C_\mu C_b{}^{;b} = 0, \tag{21}$$

$$(\mu + h)\dot{u}_a + h^b{}_a h^c{}_b - h^b{}_a C_b C_d{}^{;d} = 0. \tag{22}$$

There is a difficulty here, if we require that the "C" field

should not produce acceleration or, in other words, that the matter created should have the same velocity as the matter already in existence. We must then have

$$h^b{}_a C_b = 0. \tag{23}$$

However since C is a scalar, this implies that the rotation of the medium is zero. On the other hand if (23) does not hold, the equations are indeterminate (c.f. Raychaudhuri and Bannerjee (11)). In order to have a determinate set of equations we will adopt (23) but drop the requirement that C_a is the gradient of a scalar. The condition (23) is not very satisfactory but it is difficult to think of one more satisfactory. Hoyle and Narlikar (12) seek to avoid this difficulty by taking a particle rather than a fluid picture. However this has a serious drawback since it leads to infinite fields (Hawking (13)).

From (17),

$$C_a = -u_a \left[1 - \frac{\dot{\mu}}{\mu + \dot{\mu} + (\mu + h)\theta} \right].$$

$$\therefore \quad C^a{}_{;a} = -(\dot{\mu} + h) - (\mu + h)\theta,$$

$$= -\theta\left(1 - \frac{\dot\mu}{\mu + \dot\mu + (\mu+h)\theta}\right) + \frac{\dot\mu}{\mu + \dot\mu + (\mu+h)\theta}.$$

For $\mu \gg h$

$$(\dot\mu + \dot h) = \theta\big(1 - (\mu + h)\big),$$

$$\therefore \quad (\mu + h) \to 1.$$

Thus, small perturbations of density die away. Moreover equation (18) still holds, and therefore rotational perturbations also die away. Equation (19) now becomes

$$\dot\theta = -\tfrac{1}{3}\theta^2 - \tfrac{1}{2}(\mu + 3h) + 1,$$

$$\therefore \quad \theta \to \sqrt{3\left(\tfrac{1}{2} - h\right)}.$$

These results confirm those obtained by Hoyle and Narlikar (14). We see therefore that galaxies cannot be formed in the steady-state universe by the growth of small perturbations. However this does not exclude the possibility that there might be a self-perpetuating system of finite perturbations which could produce galaxies. (Sciama (15), Roxburgh and Saffman (16)).

8. Gravitational Waves

We now consider perturbations of the Weyl tensor that do not arise from rotational or density perturbations, that is,

$$E_{ab}{}^{;b} = H_{ab}{}^{;b} = 0$$

Multiplying (15) by $u^c\nabla_c$, and (16) by $h^a{}_{(c}\eta_{b)}{}^{rs}u_r\nabla_s$, we obtain, after a lot of reduction,

$$\ddot E_{ab} - \left(E_{cd;e}\, h^c{}_f h^d{}_g h^e{}_{(a}\, h^{fg}{}_{b)}\right) + h^{kl} h_a{}^t h_b{}^s\, \tfrac{7}{3}\, \dot E_{ab}\,\theta$$
$$+E_{ab}\left(\dot\theta + \tfrac{4}{3}\theta^2 + \tfrac{1}{3}(\mu + 3h)\right) + \sigma_{ab}\left(\tfrac{1}{3}\theta(\mu + h) + \tfrac{1}{2}(\dot\mu + \dot h)\right) = 0. \qquad (24)$$

In empty space with a non-expanding congruence u^a this reduces to the usual form of the linearised theory,

$$\Box^* E_{ab} = 0.$$

The second term in (24) is the Laplacian in the hypersurface $\tau =$ constant, acting on E_{ab}. We will write E_{ab} as a sum of eigenfunctions of this operator,

$$E_{ab} = \sum A^{(n)} V^{(n)}_{ab},$$

where

$$\dot{V}^{(n)}_{ab} = 0, \qquad \left(V_{cd;e}\, h^c{}_f h^d{}_g h^e{}_{(a} h^{fg}{}_{b)}\right) = -\frac{n^2}{\Omega^2}\, V^{(n)}_{ab},$$

$$V_{ab}{}^{;b} = 0, \qquad V_a{}^a = 0.$$

Then

$$\dot{E}_{ab} = \sum \frac{A^{(n)}}{\Omega}\, V^{(n)}_{ab}, \qquad \ddot{E}_{ab} = \sum \left[\frac{\dot{A}^{(n)}}{\Omega^2} - \frac{A^{(n)}}{\Omega^2}\right] V^{(n)}_{ab}.$$

Similarly,

$$\sigma_{ab} = \sum D^{(n)} V^{(n)}_{ab}.$$

Then by (19)

$$D'^{(n)} = \Omega A^{(n)} - 2D^{(n)}\frac{\Omega'}{\Omega}.$$

Substituting in (24),

$$A''^{(n)} + \frac{6\Omega'}{\Omega} A'^{(n)} + A^{(n)}\left[n^2 + 3\frac{\Omega''}{\Omega} + 6\frac{\Omega'^2}{\Omega^2} + \tfrac{1}{3}(\mu + 3h)\Omega^2\right] + D^{(n)}\left[\Omega(\mu + h) + \tfrac{1}{2}\Omega^2(\dot{\mu} + \dot{h})\right] = 0.$$

We may differentiate again and substitute for D'. For $n \gg 1$ and $\Omega \gg \frac{n^2}{h^2}$,

$$A^{(n)} \simeq \frac{1}{\Omega^{3/2}} e^{int}.$$

So the gravitational field E_{ab} decreases as Ω^{-1} and the "energy" $\frac{1}{2}(E_{ab}E^{ab} + H_{ab}H^{ab})$ as Ω^{-6}. We might expect this as the Bianchi identities may be written, to the linear approximation,

$$\Omega\, g^{ed} \frac{\partial}{\partial x^e}\left(\Omega^2 C_{abcd}\right) = J_{abc}.$$

Therefore if the interaction with the matter could be neglected C_{abcd} would be proportional to Ω and E_{ab}, H_{ab} to Ω^{-1}.

In the steady-state universe when μ and θ have reached their equilibrium values,

$$R_{ab} = \left(\tfrac{1}{2}\mu + h\right) g_{ab},$$

$$J_{abc} = R_{c[a;b]} - \tfrac{1}{6} g_{c[a} R_{;b]} = 0.$$

Thus the interaction of the "C" field with gravitational radiation is equal and opposite to that of the matter. There is then no net interaction, and E_{ab} and H_{ab} decrease as Ω^{-1}.

The "energy" $\frac{1}{2}(E_{ab}E^{ab} + H_{ab}H^{ab})$ depends on second derivatives of the metric. It is therefore proportional to the frequency squared times the energy as measured by the energy momentum pseudo-tensor, in a local co-moving Cartesian coordinate system, which depends only on first derivatives. Since the frequency will be inversely proportional to Ω, the energy measured by the pseudo-tensor will be proportional to Ω^{-4} as for other rest mass zero fields.

9. Absorption of Gravitational Waves

As we have seen, gravitational waves are not absorbed by a perfect fluid. Suppose however there is a small amount of viscosity. We may represent this by the addition of a term $\lambda\sigma_{ab}$ to the energy–momentum tensor, where λ is the coefficient of viscosity (Ehlers (17)).

Since

$$T_{ab}{}^{;b} = 0$$

we have

$$\dot{\mu} + (\mu + h)\theta - 2\lambda\sigma^2 = 0, \tag{25}$$

$$(\mu + h)\dot{u}_a + h^b{}_a\mu_{,b} + \lambda\sigma_{cb}{}^{;b}h^c{}_a = 0. \tag{26}$$

Equations (15) (16) become

$$\dot{E}_{ab} + E_{c(a}\theta + h^c{}_{(a}\eta_{b)de}u^c H_f{}^{d;e} = -\tfrac{1}{2}(\mu + h)\sigma_{ab} - \tfrac{1}{2}\lambda\left(E_{ab} - \tfrac{1}{3}\sigma_{ab}\theta\right), \tag{27}$$

$$\dot{H}_{ab} + H_{c(a}\theta - h^c{}_{(a}\eta_{b)de}u^c E_f{}^{d;e} = -\tfrac{1}{2}\lambda H_{ab}. \tag{28}$$

The extra terms on the right of equations (27), (28) are similar to conduction terms in Maxwell's equations and will cause the wave to decrease by a factor $e^{-\frac{1}{2}\lambda t}$. Neglecting expansion for the moment, suppose we have a wave of the form,

$$E_{ab} = E^0_{ab}e^{i\nu\tau}.$$

This will be absorbed in a characteristic time $2/\lambda$ independent of frequency. By (25) the rate of gain of rest mass energy of the matter will be $2\lambda\sigma^2$ which by (19) will be $2\lambda E^2\nu^{-2}$. Thus the available energy in the wave is $4E^2\nu^2$. This confirms that the density of available energy of gravitational radiation will decrease as Ω^{-4} in an expanding universe. From this we see that gravitational radiation behaves in much the same way as other radiation fields. In the early stages of an evolutionary universe when the temperature was very high we might expect an equilibrium to be set up between black-body electromagnetic radiation and black-body gravitational radiation. Since they both have two polarisations their energy densities should be equal. As the universe expanded they would both cool adiabatically at the same rate. As we know the temperature of black-body extragalactic electromagnetic radiation is less than 5°K, the temperature of the black-body gravitational radiation must be also less than this which would be absolutely undetectable. Now the energy of gravitational radiation does not contribute to the ordinary energy–momentum tensor T_{ab}. Nevertheless it will have an active gravitational effect. By the expansion equation,

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 - 2\sigma^2 - \tfrac{1}{2}(\mu + 3h).$$

For incoherent gravitational radiation at frequency ν,

$$\sigma^2 = \alpha E^2 \nu^{-2}.$$

But the energy density of the radiation is

$$4\, E^2 \nu^{-2}.$$

$$\therefore \quad \dot{\theta} = -\tfrac{1}{3}\theta^2 - \tfrac{1}{2}\mu_G - \tfrac{1}{2}(\mu + 3h),$$

where μ_G is the gravitational "energy" density. Thus gravitational radiation has an active attractive gravitational effect. It is interesting that this seems to be just half that of electromagnetic radiation.

It has been suggested by Hogarth (18) and Hoyle and Narlikar (10), that there may be a connection between the absorption of radiation and the Arrow of Time. Thus in universes like the steady-state, in which all electromagnetic radiation emitted is eventually absorbed by other matter, the Absorber theory would predict retarded solutions of the Maxwell equations while in evolutionary universes in which electromagnetic radiation is not completely absorbed it would predict advanced solutions. Similarly, if one accepted this theory, one would expect retarded solutions of the Einstein equations if and only if all gravitational radiation emitted is eventually absorbed by other matter. Clearly this is so for the steady–state universe since λ will be constant. In evolutionary universes λ will be a function of time. We will obtain complete absorption if $\int \lambda d\tau$ diverges. Now for a gas, $\lambda \propto T^2$ where T is the temperature. For a monatomic gas, $T \propto \Omega^{-2}$, therefore the integral will diverge (just). However the expression used for viscosity assumed that the mean free path of the atoms was small compared to the scale of the disturbance. Since the mean free path $\propto \mu^{-1}\kappa\Omega^{-3}$ and the wavelength $\propto \Omega^{-1}$, the mean free path will eventually be greater than the wavelength and so the effective viscosity will decrease more rapidly than Ω^{-1}. Thus there will not be complete absorption and the theory would not predict retarded solutions.

However this is slightly academic since gravitational radiation has not yet been detected, let alone investigated to see whether it corresponds to a retarded or advanced solution.

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Chapter 3
Gravitational Radiation In An Expanding Universe

Gravitational radiation in empty asymptotically flat space has been examined by means of asymptotic expansions by a number of authors. (1–4) They find that the different components of the outgoing radiation field "peel off", that is, they go as different powers of the affine radial distance. If one wishes to investigate how this behaviour is modified by the presence of matter, one is faced with a difficulty that does not arise in the case of, say, electromagnetic radiation in matter. For this one can consider the radiation travelling through an infinite uniform medium that is static apart from the disturbance created by the radiation. In the case of gravitational radiation this is not possible. For, if the medium were initially static, its own self–gravitation would cause it to contract in on itself and it would cease to be static. Hence one is forced to investigate gravitational radiation in matter that is either contracting or expanding.

As in Chapter 2, we identify the Weyl or conformal tensor C_{abcd} with the free gravitational field and the Ricci–tensor R_{ab} with the contribution of the matter to the curvature. Instead of considering gravitational radiation in asymptotically flat space, that is, space that approaches flat space at large radial distances, we consider it in asymptotically conformally flat space. As it is only conformally flat, the Ricci–tensor and the density of matter need not be zero.

To avoid essentially non–gravitational phenomena such as sound waves, we will consider gravitational radiation travelling through dust. It was shown in Chapter 2 that a conformally flat universe filled with dust must have one of the metrics:

(a)

$$ds^2 = -\Omega^2 \left(d\iota^2 - d\varrho^2 - \sin^2\varrho \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right), \qquad \Omega = A(1-\cos\tau) \tag{1.1}$$

(b)

$$ds^2 = \Omega^2 \left(dt^2 - d\varrho^2 - \varrho^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right), \qquad \Omega = \tfrac{1}{6}At^2 \tag{1.2}$$

(c)

$$ds^2 = \Omega^2 \left(dt^2 - d\varrho^2 - \sinh^2 \varrho \left(d\theta^2 + \sin^2 \theta \, d\phi^2\right)\right), \qquad \Omega = A(\cosh t - 1) \tag{1.3}$$

Type (a) represents a universe in which the matter expands from the initial singularity with insufficient energy to reach infinity and so falls back again to another singularity. It is therefore unsuitable for a discussion of gravitational radiation by a method of asymptotic expansions since one cannot get an infinite distance from the source.

Type (b) is the Einstein–de Sitter universe in which the matter has just sufficient energy to reach infinity. It is thus a special case. D. Norman (5) has investigated the "peeling off" behaviour in this case using Penrose's conformal technique (6). He was however forced to make certain assumptions about the movement of the matter which will be shown to be false. Moreover, he was misled by the special nature of the Einstein–de Sitter universe in which affine and luminosity distances differ. Another reason for not considering radiation in the Einstein–de Sitter universe is that it is unstable. The passage of a gravitational wave will cause it to contract again eventually and develop a singularity.

We will therefore consider radiation in a universe of type (c) which corresponds to the general case where the matter is expanding with more than enough energy to avoid contracting again.

2. The Newman–Penrose Formalism

We employ the notation of Newman and Penrose. (3) A tetrad of null vectors, $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$ is introduced where:

$$l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = l_\mu m^\mu = l_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0,$$

$$l_\mu n^\mu = 1, \qquad m_\mu \bar{m}^\mu = -1.$$

We label these vectors with a tetrad index

$$Z_a^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu), \qquad a = 1, 2, 3, 4.$$

Tetrad indices are raised and lowered with the metric

$$\eta_{ab} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{2.1}$$

We have,

$$g^{\mu\nu} = \eta^{ab} Z_a^\mu Z_b^\nu = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu. \tag{2.2}$$

Ricci rotation coefficients are defined by:

$$\gamma_{abc} = Z^\mu_{a;\nu} Z_b^\nu Z_{c\mu}, \tag{2.3}$$

$$\gamma_{abc} = -\gamma_{acb}.$$

In fact it is more convenient to work in terms of twelve complex combinations of rotation coefficients defined as follows:

$$\begin{gathered}
\kappa = \gamma_{131} = l_{\mu;\nu} m^\mu l^\nu, \\
\pi = -\gamma_{241} = -n_{\mu;\nu} \bar{m}^\mu l^\nu, \\
\epsilon = \tfrac{1}{2}(\gamma_{121} + \gamma_{341}) = \tfrac{1}{2}(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu} \bar{m}^\mu l^\nu), \\
\rho = \gamma_{134} = l_{\mu;\nu} m^\mu \bar{m}^\nu, \\
\lambda = -\gamma_{244} = -n_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu, \\
\alpha = \tfrac{1}{2}(\gamma_{124} - \gamma_{344}) = \tfrac{1}{2}(l_{\mu;\nu} n^\mu \bar{m}^\nu - m_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu), \\
\beta = \tfrac{1}{2}(\gamma_{123} - \gamma_{343}) = \tfrac{1}{2}(l_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu} \bar{m}^\mu m^\nu), \\
\sigma = \gamma_{133} = l_{\mu;\nu} m^\mu m^\nu, \\
\mu = -\gamma_{243} = -n_{\mu;\nu} \bar{m}^\mu m^\nu, \\
\nu = -\gamma_{242} = -n_{\mu;\nu} n^\mu \bar{m}^\nu, \\
\gamma = \tfrac{1}{2}(\gamma_{122} - \gamma_{342}) = \tfrac{1}{2}(l_{\mu;\nu} n^\mu n^\nu - m_{\mu;\nu} \bar{m}^\mu n^\nu), \\
\tau = \gamma_{132} = l_{\mu;\nu} m^\mu n^\nu.
\end{gathered} \tag{2.4}$$

3. Coordinates

Like Newman and Penrose, we introduce a null coordinate

$$u(=X^1), \qquad g^{\mu\nu}u_{,\mu}u_{,\nu}=0. \tag{3.1}$$

We take

$$l_\mu = u_{,\mu}.$$

Thus l_μ will be geodesic and irrotational. This implies

$$\kappa=0, \quad \rho=\bar\rho, \quad \epsilon=\bar\epsilon, \quad \alpha+\bar\beta=0. \tag{3.2}$$

We take $n^\mu, m^\mu, \bar m^\mu$ to be parallely transported along l^μ. This gives

$$\pi=\epsilon=0. \tag{3.3}$$

As a second coordinate we take an affine parameter $\gamma(=X^2)$ along the geodesics l^μ,

$$\gamma_{,\mu}l^\mu=1. \tag{3.4}$$

X^3 and X^4 are two coordinates that label the geodesic in the surface $u=$ const.

$$g^{\mu\nu}{X^i}_{,\mu}u_{,\nu}={X^i}_{,\mu}l^\mu=0. \tag{3.5}$$

Thus,

$$g^{\mu\nu}=\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & g^{23} & g^{24}\\ 0 & g^{23} & g^{33} & g^{34}\\ 0 & g^{24} & g^{34} & g^{44}\end{bmatrix}. \tag{3.6}$$

In these coordinates

$$l_\mu={\delta^1}_\mu, \qquad l^\mu={\delta^\mu}_2,$$

since $l_\mu l^\mu=0$, $l_\mu m^\mu=0$,

$$m^\mu=\cos\theta\,{\delta^\mu}_3+\sin\theta\,{\delta^\mu}_4,$$

$$n^\mu={\delta^\mu}_1+U{\delta^\mu}_2+X^i{\delta^\mu}_i, \qquad i=3,4. \tag{3.7}$$

The Field Equations

We may calculate the Ricci and Weyl tensor components from the relations

$$R_{abcd} - R_{abdc} = \gamma_{ab[c;d]} - \gamma_{ab[d;c]} + \gamma_{aeb}\gamma^{e}{}_{cd} - \gamma_{aed}\gamma^{e}{}_{bc}, \tag{3.8}$$

$$R_{abcd} = C_{abcd} + \eta_{a[c}R_{d]b} + \eta_{b[d}R_{c]a} + \tfrac{R}{3}\eta_{a[c}\eta_{d]b}. \tag{3.9}$$

Using the combinations of rotation coefficients already defined and with $\kappa = \pi = \epsilon = 0$, we have

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00}, \tag{3.10}$$

$$D\sigma = 2\rho\sigma + \Psi_0, \tag{3.11}$$

$$D\tau = \rho\tau + \bar{\sigma}\pi + \Psi_1 + \Phi_{01}, \tag{3.12}$$

$$D\alpha = \rho\alpha + \beta\bar{\sigma} + \Phi_{10}, \tag{3.13}$$

$$D\beta = \rho\beta + \alpha\sigma + \Psi_1, \tag{3.14}$$

$$D\gamma = \tau\alpha + \bar{\tau}\beta - \Psi_2 - \Lambda + \Phi_{11}, \tag{3.15}$$

$$D\lambda = \rho\lambda + \mu\bar{\sigma} + \Phi_{20}, \tag{3.16}$$

$$D\mu = \rho\mu + \sigma\lambda + \Psi_2 + 2\Lambda, \tag{3.17}$$

$$D\nu = \tau\lambda + \bar{\tau}\mu + \Psi_3 + \Phi_{21}. \tag{3.18}$$

$$\Delta\nu - \bar{\delta}\gamma = 2\alpha\nu + (\bar{\gamma} - 3\gamma - \mu - \bar{\mu})\lambda - \Psi_4, \tag{3.19}$$

$$\delta\rho - \bar{\delta}\sigma = (\bar{\beta} + \alpha)\rho + (\bar{\beta} - 3\alpha)\sigma - \Psi_1 + \Phi_{01}, \tag{3.20}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma - 2\alpha\beta + \alpha\bar{\alpha} + \beta\bar{\beta} - \Psi_2 + \Lambda + \Phi_{11}, \tag{3.21}$$

$$\delta\lambda - \bar{\delta}\mu = (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21}, \tag{3.22}$$

$$\delta\nu - \Delta\mu = \gamma\mu - 2\beta\tau + \bar{\gamma}\pi + \mu^2 + \lambda\bar{\lambda} + \Phi_{22}, \tag{3.23}$$

$$\delta\gamma - \Delta\beta = \gamma\mu - \sigma\nu - (\mu - \gamma - \bar{\gamma})\beta + \lambda\alpha + \Phi_{12}, \tag{3.24}$$

$$\delta\tau - \Delta\sigma = 2\beta\tau + (\bar{\gamma} + \mu - 3\gamma)\sigma - \bar{\lambda}\rho + \Phi_{02}, \tag{3.25}$$

$$\Delta\rho - \bar{\delta}\tau = (\gamma + \bar{\gamma} - \bar{\mu})\rho - 2\alpha\tau - \lambda\sigma - \Psi_2 - 2\Lambda, \tag{3.26}$$

$$\Delta\alpha - \delta\gamma = \rho\nu - \tau\lambda - \lambda\beta + (\bar{\gamma} - \gamma - \bar{\mu})\alpha - \Psi_3. \tag{3.27}$$

$$D = L^\mu \nabla_\mu = \frac{\partial}{\partial r} \tag{3.28}$$

$$\Delta = N^\mu \nabla_\mu = V\frac{\partial}{\partial r} + \frac{\partial}{\partial u} + X^i \frac{\partial}{\partial x^i} \tag{3.29}$$

$$\delta = M^\mu \nabla_\mu = W\frac{\partial}{\partial r} + \xi^i \frac{\partial}{\partial x^i} \tag{3.30}$$

$$\Phi_{00} = -\tfrac{1}{2}R_{11} = \overline{\Phi}_{00} \tag{3.31}$$

$$\Phi_{11} = -\tfrac{1}{4}(R_{12} + R_{34}) = \overline{\Phi}_{11} \tag{3.32}$$

$$\Phi_{01} = -\tfrac{1}{2}R_{13} = \overline{\Phi}_{10} \tag{3.33}$$

$$\Phi_{12} = -\tfrac{1}{2}R_{23} = \overline{\Phi}_{21} \tag{3.34}$$

$$\Phi_{02} = -\tfrac{1}{2}R_{33} = \overline{\Phi}_{20} \tag{3.35}$$

$$\Phi_{22} = -\tfrac{1}{2}R_{22} = \overline{\Phi}_{22} \tag{3.36}$$

$$\Lambda = \tfrac{R}{24} \tag{3.37}$$

$$\Psi_0 = -C_{1313} = -C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma m^\delta \tag{3.38}$$

$$\Psi_1 = -C_{1213} = -C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta l^\gamma m^\delta \tag{3.39}$$

$$\Psi_2 = -\tfrac{1}{2}\left(C_{1212} + C_{1234}\right) = -\tfrac{1}{2}C_{\alpha\beta\gamma\delta}\left(l^\alpha n^\beta l^\gamma n^\delta + l^\alpha n^\beta m^\gamma \bar{m}^\delta\right) \tag{3.40}$$

$$\Psi_3 = C_{1224} = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta \tag{3.41}$$

$$\Psi_4 = -C_{2424} = -C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta \tag{3.42}$$

Expressing the rotation coefficients in terms of the metric, we have:

$$D\,\xi^i = \rho\,\xi^i + \sigma\,\bar{\xi}^i \tag{3.43}$$

$$D\,\omega = \rho\,\omega + \sigma\,\bar{\omega} - (\alpha + \bar{\beta}) \tag{3.44}$$

$$D\,\chi = \tau\,\xi^i + \bar{\tau}\,\bar{\xi}^i \tag{3.45}$$

$$D\,U = \tau\,\omega + \bar{\tau}\,\bar{\omega} - (\gamma + \bar{\gamma}) \tag{3.46}$$

$$\delta\chi^i - \Delta\xi^i = (\mu + \bar{\gamma} - \gamma)\,\xi^i + \lambda\,\bar{\xi}^i \tag{3.47}$$

$$\delta\,\bar{\xi}^i - \bar{\delta}\,\xi^i = (\bar{\beta} - \alpha)\,\xi^i + (\bar{\alpha} - \beta)\,\bar{\xi}^i \tag{3.48}$$

$$\delta\,\bar{\omega} - \bar{\delta}\,\omega = (\bar{\beta} - \alpha)\,\omega + (\bar{\alpha} - \beta)\,\bar{\omega} + (\mu - \bar{\mu}) \tag{3.49}$$

$$\delta U - \Delta\omega = (\mu + \bar{\gamma} - \gamma)\,\omega + \bar{\lambda}\,\bar{\omega} - \nu \tag{3.50}$$

As in Chapter 2 we use the Bianchi identities as field equations for the Weyl tensor. In the Newman–Penrose formalism they may be written: (I am indebted to **R. G. McLenaghan** for these)

$$
\begin{aligned}
\bar{\delta}\Psi_0 - D\Psi_1 + D\Phi_{01} - \delta\Phi_{00} &= (\pi - 4\alpha)\Psi_0 - 2(2\tau + \beta)\Phi_{00} \\
&\quad + 2\rho\Phi_{01} - 2\delta\Phi_{10}
\end{aligned} \tag{3.51}
$$

$$
\begin{aligned}
\Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} &= (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 \\
&\quad + 3\sigma\Psi_2 - \lambda\Phi_{00} - 2\beta\Phi_{01} + 2\delta\Phi_{11} - \gamma\Phi_{02}
\end{aligned} \tag{3.52}
$$

$$
\begin{aligned}
3(\bar{\delta}\Psi_1 - D\Psi_2) + 2(D\Phi_{11} - \delta\Phi_{10}) &+ \bar{\delta}\Phi_{01} - \Delta\Phi_{00} \\
&= 3\lambda\Psi_0 - 9\rho\Psi_2 + 6\alpha\Psi_1 \\
&\quad + (\mu - 2\gamma - 2\bar{\gamma})\Phi_{00} + (2\alpha + 2\bar{\pi})\Phi_{01} \\
&\quad + 2(\tau - 2\alpha)\Phi_{12} + 2\rho\Phi_{11} + 2\delta\Phi_{20} - 6\Phi_{02}
\end{aligned} \tag{3.53}
$$

$$
\begin{aligned}
3(\Delta\Psi_1 - \delta\Psi_2) + 2(\Delta\Phi_{01} - \delta\Phi_{11}) &+ (\delta\Phi_{02} - \Delta\Phi_{01}) \\
&= 3\nu\Psi_0 + 6(\gamma - \mu)\Psi_1 - 9\tau\Psi_2 + 6\sigma\Psi_3 \\
&\quad + 2(\mu - \bar{\mu} - \gamma)\Phi_{01} - 2\lambda\Phi_{10} + 2\nu\Phi_{11} \\
&\quad + (2\alpha + \bar{\pi} - 2\beta)\Phi_{02} + 2\delta\Phi_{12}
\end{aligned} \tag{3.54}
$$

$$
\begin{aligned}
3(\bar{\delta}\Psi_2 - D\Psi_3) + 2(D\Phi_{21} - \delta\Phi_{20}) &+ 2(D\Phi_{11} - \delta\Phi_{21}) \\
&= 6\mu\Psi_1 - 6\rho\Psi_3 - 2\pi\Phi_{02} + 2(\mu - \bar{\mu} - \gamma)\Phi_{10} \\
&\quad + 4\bar{\alpha}\Phi_{11} + (2\beta - 2\alpha - 2\pi)\Phi_{20} - 2\delta\Phi_{12}
\end{aligned} \tag{3.55}
$$

$$
\begin{aligned}
3(\Delta\Psi_2 - \delta\Psi_3) + (D\Phi_{22} - \delta\Phi_{21}) + 2(\bar{\delta}\Phi_{12} - \Delta\Phi_{11}) = {} & 6\nu\Psi_2 - 9\mu\Psi_2 + 6(\beta - \tau)\Psi_3 - 3\sigma\Psi_4 \\
& - 2\nu\Phi_{01} + 2\bar{\nu}\Phi_{10} + 2(\bar{\mu} - \mu)\Phi_{11} + 2\lambda\Phi_{02} \\
& - \lambda\Phi_{20} + 2(\bar{\pi} - 2\beta)\Phi_{12} + 2(\beta\tau - \gamma\mu)\Phi_{21} - 6\Phi
\end{aligned}
\tag{3.56}
$$

$$
\begin{aligned}
\bar{\delta}\Psi_3 - D\Psi_4 + \bar{\delta}\Phi_{21} - \Delta\Phi_{20} = {} & 3\lambda\Psi_2 - 2\nu\Psi_3 - \rho\Psi_4 \\
& - 2\nu\Phi_{01} + 2\lambda\Phi_{10} + (2\gamma - 2\bar{\gamma} - \mu)\Phi_{20} \\
& + 2(\bar{\pi} - \alpha)\Phi_{21} - 6\Phi_{22}
\end{aligned}
\tag{3.57}
$$

$$
\begin{aligned}
\Delta\Psi_3 - \delta\Psi_4 + \bar{\delta}\Phi_{22} - \Delta\Phi_{21} = {} & 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4 \\
& - 2\nu\Phi_{11} - \bar{\nu}\Phi_{00} + 2\lambda\Phi_{02} \\
& + 2(\gamma + \bar{\mu})\Phi_{11} + (\bar{\pi} - 2\beta - 2\tau)\Phi_{22}
\end{aligned}
\tag{3.58}
$$

$$
\begin{aligned}
D\Phi_{12} - \delta\Phi_{11} - \bar{\delta}\Phi_{02} + \Delta\Phi_{01} + 3\Lambda = {} & (2\gamma - \mu - 2\bar{\mu})\Phi_{01} + \pi\bar{\nu}\Phi_{00} \\
& - \lambda\Phi_{02} - 2\pi\Phi_{11} + (2\beta - 2\alpha - \bar{\pi})\Phi_{02} + 3\rho\Phi_{22} +
\end{aligned}
\tag{3.59}
$$

$$
\begin{aligned}
D\Phi_{10} - \delta\Phi_{01} + \Delta\Phi_{00} + 3D\Lambda = {} & (2\gamma - \mu + 2\bar{\gamma} - \bar{\mu})\Phi_{00} - 2(\alpha + \bar{\pi})\Phi_{10} \\
& - 2(\tau + \alpha)\Phi_{10} + 4\rho\Phi_{11} + 6\Phi_{02} - \tau\Phi_{20}
\end{aligned}
\tag{3.60}
$$

$$
\begin{aligned}
D\Phi_{22} - \delta\Phi_{21} - \bar{\delta}\Phi_{12} + \Delta\Phi_{11} + 3\Delta\Lambda = {} & \nu\Phi_{01} + \bar{\nu}\Phi_{10} - 2(\mu + \bar{\mu})\Phi_{11} \\
& - \lambda\Phi_{02} - \bar{\lambda}\Phi_{20} + (2\bar{\beta} - \tau)\Phi_{12} + (2\beta - \tau)\Phi_{21} \\
& + 2\rho\Phi_{22}
\end{aligned}
\tag{3.61}
$$

4. The Undisturbed Metric

The undisturbed metric may be written

$$ds^2 = \Omega^2 \left(dt^2 - d\rho^2 - \sinh^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

where

$$\Omega = A(\cosh t - 1).$$

Put

$$u = t - \rho,$$

then

$$ds^2 = \Omega^2 \left[-du^2 + 2\, du\, dt - \sinh^2(t-u)\, (d\theta^2 + \sin^2\theta\, d\phi^2) \right] \tag{4.1}$$

u is a null coordinate.

To calculate r, the affine parameter, we note that t is an affine parameter for the metric within the square brackets. Therefore

$$r = \int \Omega^2\, dt = B(u, \theta, \phi) \tag{4.2}$$

will be an affine parameter for (4.1).

B is constant along the null geodesic. Normally it would be taken so that $r = 0$ when $t = u$. However, in our case it will be more convenient to make it zero and define r as

$$r = \int_0^t \Omega^2\, dt' \tag{4.3}$$

This means that surfaces of constant r are surfaces of constant t. This may seem rather odd, but it should be pointed out that the choice of B will not affect the asymptotic dependence of quantities. That is, if

$$f = O(r^{-n})$$

then

$$f = O(r'^{-n}), \qquad r' = r + B.$$

It proves easier to perform the calculations with this choice of r but all results could be transformed back to a more normal coordinate system.

From (4.3)

$$r = A^2 \left[\tfrac{1}{4} \sinh 2t - 2 \sinh t + \tfrac{3}{2} t \right] \tag{4.4}$$

The matter in the universe is assumed to be dust so its energy tensor may be written

$$T_{ab} = \mu\, V_a V_b \tag{4.5}$$

For the undisturbed case, from Chapter 2

$$\mu = \frac{6A}{\Omega^3} \tag{4.6}$$

$$V_a = \Omega\, t_{;a}, \qquad V_a V^a = 1$$

Now

$$\Omega = \sqrt{2s} + A - \frac{3A^2}{2\sqrt{2}} \frac{\log s}{s} + O(s^{-1}) \tag{4.7}$$

where

$$s^2 = r.$$

Therefore if we try to expand μ as a series in powers of s the result will be very messy and will involve terms of the form

$$\frac{\log^n s}{s^n} x^{(*)}$$

** It should be pointed out that the expansions used will only be assumed to be valid asymptotically. They will not be assumed to converge at finite distances nor will the quantities concerned be assumed analytic. (see A. Erdelyi: Asymptotic Expansions – Dover)*

This does not invalidate it as an asymptotic expansion but it makes it tedious to handle. For convenience therefore, we will perform the expansions in terms of $\Omega(r)$ which will be defined in general as the same function of r as it is in the undisturbed case. That is

$$\Omega = A(\cosh t - 1), \qquad r = A^2\left[\tfrac{1}{4}\sinh 2t - 2\sinh t + \tfrac{3}{2}t\right] \tag{4.8}$$

then

$$\frac{d\Omega}{dr} = \frac{\sqrt{1+\frac{3}{2}A}}{\Omega} = \frac{1}{\Omega}\left[1 + \frac{A}{\Omega} - \frac{A^2}{2\Omega^2} + \frac{A^3}{2\Omega^3} - \frac{5A^4}{8\Omega^4} + \frac{7A^5}{8\Omega^5} - \cdots\right] \tag{4.9}$$

For the third and fourth coordinates it is more convenient to use stereographic coordinates than spherical polars.

Since the matter is dust its energy–momentum tensor and hence the Ricci tensor have only four independent components. We will take these as Λ, Φ_{00}, Φ_{01} (since Φ_{01} is complex it represents two components).

In terms of these the other components of the Ricci tensor may be expressed as:

$$\Phi_{11} = 3\Lambda + \frac{\Phi_{01}\bar{\Phi}_{01}}{\Phi_{00}}$$

$$\Phi_{22} = \frac{3}{\Phi_{00}}\Lambda^2 \left(1 + \frac{\Phi_{01}\bar{\Phi}_{01}}{6\Lambda\Phi_{00}}\right)^2$$

$$\Phi_{12} = \bar{\Phi}_{21} = \frac{6\Lambda\Phi_{01}}{\Phi_{00}} \left(1 + \frac{\Phi_{01}\bar{\Phi}_{01}}{6\Lambda\Phi_{00}}\right)^2$$

$$\Phi_{02} = \bar{\Phi}_{20} = \frac{\Phi_{01}^2}{\Phi_{00}} \tag{4.10}$$

—

For the undisturbed universe with the coordinate system given:

$$\Lambda = \frac{\mu}{24} = \frac{A}{4\Omega^3}$$

$$\Phi_{00} = \frac{3A}{\Omega^5}, \qquad \Phi_{11} = \frac{3A}{4\Omega^3}, \qquad \Phi_{22} = \frac{3A}{\Omega}$$

$$\Phi_{01} = \Phi_{02} = 0 \tag{4.11}$$

—

Using these values and the fact that in the undisturbed universe all the $\Psi^{(n)}$ are zero, we may integrate equations (3.10–50) to find the values of the spin coefficients for the unperturbed universe:

$$\rho = -\frac{2}{\Omega^2} - \frac{A}{\Omega^3} + \left(\frac{A^2}{2} - \frac{A^2}{2}e^{2u}\right)\Omega^{-4} + A^3(u - e^{2u})\Omega^{-5} + A^4\left(\tfrac{5}{8} - \tfrac{1}{4}u - \tfrac{1}{8}e^{4u}\right)\Omega^{-6} + \cdots$$

$$\epsilon = \pi = \kappa = \nu = \lambda = \chi = 0$$

$$\xi^2 = \frac{Se^{iu} - ASe^{iu}}{\Omega^2} = \frac{Se^{iu}}{\Omega^3} + \tau A^4(5SAu + Se^{3u})\Omega^{-4} + \cdots$$